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SENIOR THESIS

Extra invariants in Hamiltonian Systems

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Abstract

In this paper, we study the dynamics of physical systems when there are many invariants of motion that restrict the possible motion of the system. First we introduce the relevant concepts for the study of dynamical systems, starting from basic Newtonian physics and working up to general Hamiltonian systems. Then we study the behavior of integrable systems (the systems with what is normally the "maximum" number of invariants) when they have even more invariants. Using differential topology, we prove in the 2 degree of freedom case that the system will exhibit periodic motion and the type of periodicity is fixed: for instance a periodic system that goes around twice before returning to its original system will always exhibit such behavior, regardless of its starting conditions. Extra invariants are seen in systems such as the two-body problem of gravitational motion.

1 Introduction

1.1 Elementary Physics

A problem in elementary Newtonian physics usually consists of a particle of mass m and a force acting on this particle. This force can depend on the position and momentum of the particle. The simplest example is a particle moving in a single dimension. Then the position q of the particle at a given time is described by a real-valued function $q(t) : \mathbb{R} \to \mathbb{R}$.

By Newton's second law, the force satisfies the equation $F(p, q) = m\ddot{q}(t)$. Momentum is given by the equation $p(t) = m\dot{q}(t)$, so finding the motion of a particle is a matter of solving for the position $q(t)$ and momentum $p(t)$ using a coupled system of differential equations. These differential equations are namely:

$$\dot{q}(t) = \frac{p(t)}{m}$$
$$\dot{p}(t) = F(p, q)$$

There are then two relevant trajectories in this problem, $p(t)$ and $q(t)$. Together, they can be thought of as a single trajectory $\vec{f}(t) : \mathbb{R} \to \mathbb{R}^2$ into two-dimensional space with $\vec{f}(t) = (q(t), p(t))$. Then the physics is reduced to knowing $\frac{d\vec{f}}{dt}$ and solving for $\vec{f}(t)$.

This can be generalized to a particle moving in three dimensions by making $\vec{q}(t)$ and $\vec{p}(t)$ three-dimensional vectors. Then $\vec{f}$ is simply a six-dimensional vector, but $\frac{d\vec{f}}{dt}$ is still known, so it's simply a problem of solving a differential equation for a trajectory $\vec{f}(t)$ in $\mathbb{R}^6$.

A further extension of this concept is the motion of n particles in three dimensions. If the force $\vec{F}_i$ acting on the ith particle is a function of $\vec{q}_1, \ldots, \vec{q}_n, \vec{p}_1, \ldots, \vec{p}_n$ then once again solving for $\vec{q}_1, \ldots, \vec{q}_n, \vec{p}_1, \ldots, \vec{p}_n$ is a system of coupled differential equations. As before, the dynamics of the system can be thought of as a single vector $\vec{f} = (\vec{q}_1, \ldots, \vec{q}_n, \vec{p}_1, \ldots, \vec{p}_n)$ moving through $6n$-dimensional space $\mathbb{R}^{6n}$ and $\frac{\partial \vec{f}}{\partial t}$ is known at each point in $\mathbb{R}^{6n}$. Then solving for $\vec{f}$ is the same process as before.

In all these cases, finding the dynamics of the system is reduced to finding some path in some $2n$-dimensional space by solving a differential equation. This $2n$ dimensional space is called the phase space of the system. The differential equation defining the trajectory $f(t)$ in phase space is defined by a vector field $\frac{\partial f}{\partial t}(p)$.

1.2 Topology

In general, the phase space doesn't have to be $\mathbb{R}^{2n}$. For instance if a particle is restricted to move along a circle, then the position of the particle cannot be described by just a real number q, but instead a point on a circle.

In full generality, phase space should be thought of as a $2n$-dimensional manifold M. The exact definition of a smooth manifold is beyond the scope of this thesis, but simply put an n-dimensional manifold is a collection of points put together so that around any point there's a region of points that looks like $\mathbb{R}^n$. For example, at any point on a circle, the points nearby look like an interval on the real line. Similarly, at any point on a sphere the points nearby look like a patch of the plane. More formally, around every point p there's a neighborhood of points U such that there's a map $\phi : U \to \mathbb{R}^n$ that pairs each point in $\mathbb{R}^n$ with exactly one point in U. This puts n "coordinates" on U and allows us to do calculus in U by treating it as $\mathbb{R}^n$.

Using this concept, we can define a vector on a manifold. Defining a vector at a point in $\mathbb{R}^n$ is simple, the standard example being a vector field in $\mathbb{R}^n$. In this case, each point p in $\mathbb{R}^n$ has a vector $\vec{v}_p \in \mathbb{R}^n$ associated to it, so a vector at a point can be defined as a pair $(p, \vec{v})$ where $p \in \mathbb{R}^n$ is the point the vector is based at and $\vec{v} \in \mathbb{R}^n$ is the direction of the vector at p.

Using this definition we define a vector at a point p on M by choosing a chart $\phi : U \to \mathbb{R}^n$ around p and choosing a vector at $\phi(p) \in \mathbb{R}^n$ the way described before. To describe this same vector with a different choice of chart around the same point, simply require that the vector satisfy the standard vector transformation laws of a change of coordinates. [3]

Using manifolds and tangent vectors, we can describe dynamics taking place in more general phase spaces. Let the phase space M be a $2n$-dimensional manifold. In previous examples, we had a vector field $\frac{\partial \vec{f}}{\partial t}(p)$ at every point $p \in M$, and solved a differential equation for $\vec{f}(t)$. So in general, dynamics are given by a phase space M and a vector field V on the manifold. One way to calculate V is knowing the forces acting on the system. Another way is via the methods of Hamiltonian mechanics.

1.3 Hamiltonian Mechanics

From this point on I assume a basic knowledge of differential topology.

In the simplest case where phase space is $\mathbb{R}^2$, representing motion along a line, a Hamiltonian system is defined by the differential equations $\dot{p} = -\frac{\partial H}{\partial q}$ and $\dot{q} = \frac{\partial H}{\partial p}$. In these equations, q is the position coordinate, p is the momentum, and H is the Hamiltonian function. The Hamiltonian function H expresses the energy of the system at a given position q and momentum p.

To produce H, consider the partcle of mass m moving in a potential $V(q)$. Then the Hamiltonian (the energy) at a given p and q is $H(p,q) = \frac{p^2}{2m} + V(q)$, the sum of kinetic and potential energy.

Applying Hamilton's equations, $\dot{p} = -\frac{\partial V}{\partial q}$ and $\dot{q} = \frac{p}{m}$. Substituting $p = m\dot{q}$ yields $\dot{q} = \frac{p}{m}$ and $m\ddot{q} = -\frac{\partial V}{\partial q}$, the equations of motion of a particle in a potential.

Generalizing phase space to a $2n$ dimensional manifold M, a Hamiltonian function H needs to produce a vector field on M. More precisely, just like $(\dot{q}, \dot{p})$ depends on $\frac{\partial H}{\partial p}$ and $\frac{\partial H}{\partial q}$, the differential dH should produce a vector field on M.

The differential of a function produces a differential 1-form on the manifold M. The conversion between a vector and a 1-form can be achieved by a bilinear form on a vector space. The classic example is using a dot product on a vector space V to produce an isomorphism between V and V^*. However, any non-degenerate bilinear form ω can be used to produce an isomorphism $f : V \to V^*$ sending $v \mapsto \omega(v, \cdot)$. Using this isomorphism on the tangent space of M, a non-degenerate 2-form ω can send dH to a vector field we label $\omega^{-1}(dH)$.

In the earlier example, the 2-form that takes dH and produces $\vec{v} = (\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial q})$ is $\omega = dq \wedge dp$. To see this, note that $\omega(\vec{v}, \cdot)$ acting on an arbitrary vector $\vec{u} = (u_q, u_p)$ can be expressed as

$$\begin{aligned}
\omega((\dot{q}, \dot{p}), \vec{u}) &= \dot{q}u_p - \dot{p}u_q \\
&= \frac{\partial H}{\partial p}u_p + \frac{\partial H}{\partial q}u_q \\
&= \frac{\partial H}{\partial p}dp(\vec{u}) + \frac{\partial H}{\partial q}dq(\vec{u}) \\
&= dH(\vec{u}).
\end{aligned}$$

Thus $\omega(\vec{v}, \cdot) = dH$ so ω sends $\vec{v}$ to dH and vice versa, which we will denote as $\omega^{-1}(dH) = \vec{v}$.

For a general phase space M, we require that it be a symplectic manifold, that is a manifold with a choice of symplectic form ω. Then any function H can produce a vector field $\omega^{-1}(dH)$. Knowing this, we can define a Hamiltonian system.

Definition 1. A Hamiltonian system is an even-dimensional manifold M together with a symplectic form ω and a Hamiltonian function $H : M \to \mathbb{R}$.

1.4 Integrable Systems

To define an integrable system, we first must define a the Poisson bracket of two real-valued functions.

Definition 2. The Poisson bracket of two functions $f : M \to \mathbb{R}, g : M \to \mathbb{R}$ on a manifold with symplectic form ω is defined as $\{f, g\} = df(\omega^{-1}(dg))$

The motivation for this definition is that $\{f, H\} = df(\omega^{-1}(H)) = df(\vec{v}) = \frac{df}{dt}$. So a function f is an invariant of the evolution of the Hamiltonian system if and only if $\{f, H\} = 0$ Then in some sense the Poisson bracket of two functions is a measure of their independence, where two functions are “Poisson-independent” if $\{f, g\} = 0$.

Definition 3. An n-degree of freedom integrable dynamical system is a Hamiltonian system with n invariants of motion $f_1, \ldots, f_n$ such that $\forall i, j < n, \{f_i, f_j\} = 0$ and the differentials $\{df_1, \ldots, df_n\}$ are linearly independent. If this is satisfied, then there exist canonical coordinates (coordinates where Hamilton’s equations apply) $J_1, \ldots, J_n, \theta_1, \theta_n$ called action-angle coordinates such that the Hamiltonian H is only a function of the action coordinates $J_1, \ldots, J_n$. [1]

Hamilton's equations tell us that $\frac{\partial \theta_i}{\partial t} = \frac{\partial H}{\partial J_i}$ and $\frac{\partial J_i}{\partial t} = \frac{\partial H}{\partial \theta_i}$ Since H depends only on the action coordinates, $\frac{\partial J_i}{\partial t} = 0$ and $\omega_i := \frac{\partial \theta_i}{\partial t}$ is constant when the action coordinates are fixed. Therefore the system evolves as follows: the action angles stay constant and the angle coordinates vary at a constant velocity on a given level set of action coordinates.

Another fact about integrable systems is that a level set of the action coordinates are n-dimensional tori if the level set is compact. In this case, the angle variables will in fact be periodic, that is θ_i and $\theta_i + 2\pi$ refer to the same points. Then when studying an integrable system, the study of motion devolves into studying straight-line trajectories on the different toroidal level sets.

In this thesis, we assume an integrable system is such that every level set is such a torus. This is a very restricted definition of an integrable system and there are several other ways to define it, but for the study of 2-body celestial motion this is enough.

1.5 Torus Trajectories

For an n-dimensional torus there are n frequencies $\omega_1, \ldots, \omega_n$. Depending on the values of these frequencies, the trajectory can fill out different amounts of the torus. Take the case of the two-dimensional torus. Then there are two frequencies, ω_1 and ω_2. Suppose the angular coordinates θ_1, θ_2 are taken mod 2π, that is θ_i and $\theta_i + 2\pi$ refer to the same points. Now assume that ω_1 and ω_2 are commensurate, that is there exist integers $a, b \in \mathbb{Z}$ such that $a\omega_1 + b\omega_2 = 0$. Then after a time $\Delta t = \frac{2\pi b}{\omega_1}$ has elapsed, $\Delta\theta_1 = 2\pi b$ and $\Delta\theta_2 = 2\pi \frac{b\omega_2}{\omega_1} = 2\pi \frac{-a\omega_1}{\omega_1} = -2\pi a$. Both $\Delta\theta_i$ are integer multiples of 2π, so the point is back where it started. So the path of a point would wind around the torus several times before ending back up at the starting point. It is a circle embedded in the torus. If ω_1 and ω_2 are not rationally commensurate, it's a theorem that the path will never loop back over itself, and instead densely fill the torus. In this case, the path is not an embedded 1-dimensional manifold. [1]

2 Results

The results of this thesis deal with integrable systems that have two degrees of freedom and one extra invariant.

Theorem 1. On an integrable 2 degree-of-freedom system, the existence of an extra invariant function $W : M \to \mathbb{R}$ on the phase space M that is functionally independent from two invariants H, I implies frequency locking: the ratio $\frac{\omega_1}{\omega_2}$ is rational and constant.

Proof. Since H, I, and W are functionally independent and all invariants of the motion, then an intersection of level sets of these functions is an embedded

1-manifold. Since these functions are invariants of motion, we also know that any trajectory in the system must be contained in a 1-manifold. Switching to action-angle variables$(J_1, \theta_1, J_2, \theta_2)$, the level set fixing (J_1, J_2) is a torus. Then the frequencies ω_1, ω_2 must be commensurate in order for the trajectory to lie within an embedded 1-manifold. Therefore $\frac{\omega_1}{\omega_2}$ is rational. A priori this value may depend smoothly on the choice of j_1, j_2. However $\frac{\omega_1}{\omega_2}$ cannot be irrational and the irrational numbers are dense in the real numbers, so $\frac{\omega_1}{\omega_2}$ cannot vary. $\square$

2.1 Construction of Extra-Invariant Systems

We can then classify the the possible Hamiltonians on an integrable system that admit an extra invariant. Let $\frac{a}{b}$ be the rational number that $\frac{\omega_1}{\omega_2}$ is locked at. $H(J_1, J_2)$ must satisfy $\frac{\partial H}{\partial J_1}/\frac{\partial H}{\partial J_2} = \frac{a}{b}$, so $b\frac{\partial H}{\partial J_1} - a\frac{\partial H}{\partial J_2} = 0$. This is equivalent to the equation $\nabla H(J_1, J_2) \cdot (b, -a) = 0$, so the differential equation states that H must be constant along the lines in (J_1, J_2) space with slope $-\frac{a}{b}$.

Since H is constant along these lines, $H(J_1, J_2)$ is totally defined by value of H on the line $J_2 = 0$ so long as $b \neq 0$ in the fraction $\frac{a}{b}$. If $b = 0$, simply use any other line, such as defining H on the line $J_1 = 0$.

Now supposed H is a Hamiltonian of this form. Then by the construction of H, the trajectories of motion along a level set of (J_1, J_2) follow a straight line of slope $\frac{b}{a}$ in θ_1, θ_2 space. Thus the function $W = \sin(b\theta_1 - a\theta_2)$ is an invariant of motion since it's a function of only $a\theta_1 - b\theta_2$ and thus constant on lines of slope $\frac{a}{b}$ in θ_1, θ_2 space on a level set of (J_1, J_2). In fact, any 2π-periodic function $f(b\theta_1 - a\theta_2)$ will do. Thus we know $\{W, H\} = 0$. We also know $\{J_1, H\} = 0$ and $\{J_2, H\} = 0$ by the definition of action-angle coordinates. Since W is only defined in terms of θ_1, θ_2 and J_1 and J_2 are functions independent of these coordinates, it's clear that W is functionally independent from J_1 and J_2. Thus, a Hamiltonian H of the form constructed above admits the 2 invariants required for integrability and an extra invariant.

Thus we have classified the integrable Hamiltonian systems on the phase space M.

3 Discussion

The most immediate application of this result is to orbital mechanics. In a two-body system governed by Newtonian gravity, the resulting system is integrable with an extra invariant called the Runge-Lenz vector. In a particle of mass m under the influence of the central force $\vec{F} = -\frac{k}{r^3}\vec{r}$, this vector is given by the equation $\vec{A} = \vec{p} \times \vec{L} - mk\frac{\vec{r}}{|\vec{r}|}$, where $\vec{L}$ is the angular momentum. After solving for Kepler's equations, $\vec{A}$ turns out to be a vector pointing along the major axis of the elliptical orbit, proportional to the eccentricity of the ellipse. [2] A priori $\vec{A}$ gives three invariants of motion correspond to the components A_x, A_y, A_z but together these are not completely linearly independent from the 2 invariants needed to prove integrability of the system. In effect, A_x, A_y, A_z

provide a single invariant. Thus orbital mechanics exhibits frequency locking. This prediction is verified by Keplerian motion. For closed orbits, an object always orbits in a circle or an ellipse. The motion is periodic and frequency-locked since all the orbits require exactly one go-around to return to their initial configuration.

The existence of an extra invariant doesn't guarantee frequency locking when there are more than two degrees of freedom. In the case of three degrees of freedom, having a single extra invariant doesn't guarantee frequency locking. Namely, for the three frequencies $\omega_1, \omega_2, \omega_3$, the ratios formed between some two frequencies might not be fixed. Having one extra invariant guarantees that there exist $a, b, c \in \mathbb{Z}$ such that $a\omega_1 + b\omega_2 + c\omega_3 = 0$ and a, b, c are not all identically zero. Suppose $c = 0$. Then ω_3 can vary in any way, and the equation will still be true. This means ω_1/ω_3 can vary, for instance.

To achieve a similar phase locking, another invariant is required. This requires the trajectory to lie on a 1-manifold, which is equivalent to requiring $\omega_1, \omega_2, \omega_3$ to have some $\lambda > 0$ such that the $\lambda\omega_i$ are all integers. So a triple of frequencies is valid if the line the vector $(\omega_1, \omega_2, \omega_3)$ describes in $\mathbb{R}^3$ passes through a point on the integer lattice. However, there are only countably many points on the integer lattice so as we vary the action coordinates (J_1, J_2, J_3), the line in $\mathbb{R}^3$ described by the frequencies must be constant since it must vary continuously while also always passing through a point on the integer lattice. Thus the system exhibits frequency locking, since $(\omega_1, \omega_2, \omega_3)$ can only vary by scalar multiplication.

This argument can be easily generalized to n degrees of freedom, where $n - 1$ additional invariants are required to prove frequency locking. This bears a resemblance to reduction of order [2]. Further work can be done in explicitly writing out how frequency locking is implied from this theorem.

Another potential avenue of exploration is studying what extra invariants in higher degree of freedom systems imply. It may not necessarily give frequency locking, but there's reason to believe that extra invariants still put some strong restrictions on the possible frequencies. For instance in the integrable 3 degree-of-freedom case with one extra invariant, the line described by the vector $(\omega_1, \omega_2, \omega_3)$ must pass through the set of "coordinate lines": the set of all points in $\mathbb{R}^3$ with two integer coordinates. Unlike with when the line had to cross the integer lattice, it's now possible for the slope of the line to vary as (J_1, J_2, J_3) varies. There may yet be more restrictions on the possible values of the frequency vector $(\omega_1, \omega_2, \omega_3)$. Keeping track of the first point on the "coordinate lines" that the line defined by $(\omega_1, \omega_2, \omega_3)$ intersects, as (J_1, J_2, J_3) vary this point moves on the "coordinates lines". However it needs to take a right-angle turn to be able to get to some points. Taking a sharp turn isn't a smooth trajectory, so it may not be possible to reach all points on the coordinate lines. If proven true, this could be generalized to a more general form of frequency locking.

Another possible extension of this work is to do this on a more generalized type of integrable system. In this thesis it was assumed that phase space could essentially be decomposed into $\mathbb{R}^2 \times \mathbb{T}^2$. In more general integrable systems this doesn't hold true, and it would be interesting to study how singular fibers in the space could affect phase locking.

References

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