QUANTUM CONTROL

(Term paper – PHY 392T)

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In this term paper, we review the theory and applications of quantum (or coherent) control. Classical control theory has been developed and used extensively by engineers and applied mathematicians for much of the last century. The program of applying the concepts of control theory to quantum systems is relatively recent. Here, we will give a brief introduction to classical control theory, followed by a non-rigorous exposition of the theory of quantum control. Then, we will look at how these concepts are applied to manipulate the dynamics of a variety of quantum systems, chemical reactions, and, perhaps most pertinently for us, semiconductor nanostructures.

I. INTRODUCTION

The study of classical control systems has been ubiquitous across engineering disciplines for most of the last century. It has been studied and developed for two major reasons. The first is to develop the abstract mathematical tools, viz. *control theory*, to describe and understand dynamical systems. The second is to precisely control the evolution of systems in a noisy environment, usually via *feedback*. Examples where control theory has been successfully applied include airplane pitch and yaw control, chemical reactors, precision electrical circuits, and limiting vibrations in mechanical systems. We shall start by looking at some of the basic concepts of this incredibly successful theory, in its simplest form: *linear, time-invariant (LTI)*.

The rapid evolution of electronic and mechanical systems toward the nanometer scale makes it attractive to develop and study a control theory that might be applicable to these systems. This interest started in the 1980's, and gathered steam after a series of landmark papers [1] describing a scheme to control the dynamics of a laser cavity by continuous feedback via homodyne detection (i.e. detection of a signal using a local oscillator of the same frequency). We shall review the basics of the theory of quantum feedback control [2], pointing out how the peculiarities of quantum mechanical systems and measurements performed thereupon inform this theory.

Finally, we shall look at a few representative proposed or realized applications of quantum control theory. These will include coherent control of chemical reactions and carrier dynamics in nanostructures. The latter provides a hint to the deep connection between quantum control and quantum computing/information. In fact, it is now expected that robust quantum computing and/or information processing will require use of the concepts of quantum control.

II. CLASSICAL CONTROL

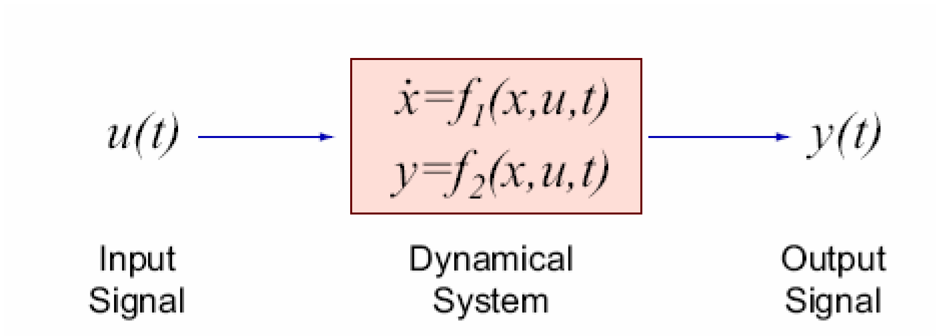
In this section, and the next, the mathematical developments will follow the exposition of Geremia [2]. Suppose the vector of state variables of the controlled classical dynamical LTI system $\mathbf{x}(t) \in \mathbb{R}^n$, explicitly $\{x_1(t), x_2(t), \ldots, x_n(t)\}$, evolves according to the equation of motion:

$$\begin{aligned} \frac{d}{dt}\mathbf{x}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{aligned}$$

where $\mathbf{u}(t)$ is the vector of input control signals, $\mathbf{y}(t)$ is the vector of observed output signals, and the matrices **A, B, C, D** are time-invariant. From the standard theory of differential equations,

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 + e^{\mathbf{A}t}\int_0^t e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

is the solution to above equation of motion. Schematically, the system looks as follows:



Classical control systems might be (i) open loop, or (ii) closed loop (or feedback). In the first case, the optimal control signal is designed from a priori knowledge of the system model. Obviously, this method might not be robust against disturbances, as well as errors in the system model used. In the second case, the measurement $\mathbf{y}(t)$ is used to tune the control signal in real time. It is compared to a desired reference output signal $\mathbf{r}(t)$ and the difference is used to generate a control signal $\mathbf{u}(t)$ that will tend to close the gap. As might be expected, most real-life control systems utilize feedback.

If the feedback system that generates $\mathbf{u}(t)$ from $\mathbf{y}(t)$ has an impulse response (Green's function) $F(t)$, then it follows that

$$u(t) = \int_0^t F(t-\tau)y(\tau)\,d\tau \tag{1}$$

Most of control system engineering involves the design of an optimal $F(t)$ that will ensure precision and stability. While we cannot go into the details of classical control any more than we have for reasons of brevity and time, suffice it to say that there exist rigorous mathematical techniques to carry out the aforementioned design [3].

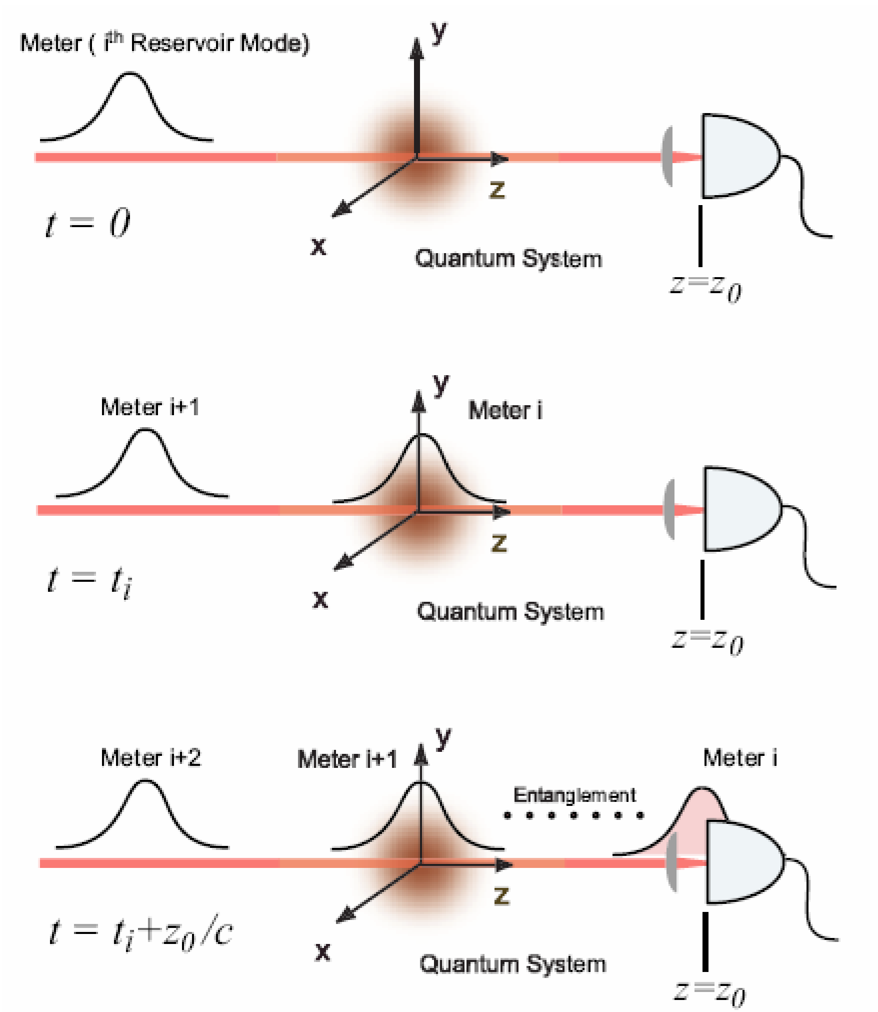
III. QUANTUM CONTROL THEORY

We have seen, for the case of classical control, that continuous observation of the system is essential to designing the control protocol. This presents us with an apparent problem in the case of quantum mechanical systems, if observation is to entail a projective measurement of its state. Fortunately, one can carry out what are called weak measurements [2], in which the system only gradually collapses to one of the eigenstates of the measurement operator. These measurements can be performed continuously and therefore it on these that quantum feedback control theory is built. The formal tool used to study quantum systems under continuous feedback is called quantum trajectory theory [4] (A more abstract mathematical method of formulating the quantum control problem is in terms of the Lie algebra generated by the Hamiltonian and control operators [5]; in fact, this approach elucidates the fact that quantum control, in the sense of implementing prescribed unitary transformations in state space is intimately tied to quantum computing and/or information processing). We will look at the physical picture emerging from this theory, in the simplest possible terms.

First, it might be useful to point out the similarities and correspondences between the classical and quantum control theory. The linear version of classical theory mentioned here, of course, shares the linearity property with quantum systems. The state space in the classical case corresponds to a (possibly infinite dimensional) Hilbert space in the quantum case. The state vector corresponds to the density matrix $\hat{\rho}(t)$ in the quantum system. The system matrix **A** is analogous to the Liouvillian, which propagates the density matrix as follows: $d\,\hat{\rho}(t) = \mathcal{L}_0\hat{\rho}(t)\,dt$. In the special case where the Liouvillian is a Hamiltonian, the equation of motion for the system is the von Neumann equation: $\frac{d}{dt}\hat{\rho} = -i\hbar[\hat{H}_0, \hat{\rho}]$. The forcing term $\mathbf{Bu}(t)$ corresponds, in the quantum case, to the interaction Hamiltonian $u(t)\hat{H}_{\mathrm{fb}}$, which adds a feedback control term to the system Liouvillian: $d\hat{\rho}(t) = \mathcal{L}_0\hat{\rho}(t)\,dt - i\hbar u(t)[\hat{H}_{\mathrm{fb}}, \hat{\rho}]$. Physically this arises from the interaction of the system and the external control field, usually from a laser. Lastly, the observation process in the classical system corresponds to conditional evolution in the quantum case, where it is well known that observing the system will affect the evolution in the form of backaction noise.

The way to carry out a weak (non-projective) measurement on a quantum system is via what is called an indirect measurement [6]. This involves entangling the system with a disposable meter (e.g. laser pulse) first, switching off the interaction between the meter and system and then carrying out a projective measurement on the meter. Though the quantum system is never actually perturbed by the measurement, it still encounters (Heisenberg limited) backaction noise because of the entanglement.

Following [2], let us suppose the system belongs to a Hilbert space $\mathcal{H}_S$, and the meters R_i to a Hilbert space $\mathcal{H}_R$ for the reservoir (collection of meters). The scheme is illustrated below:



The initial density operator for the composite system (system and reservoir) can be written as a tensor product of the system density operator and the density operator for the reservoir mode respectively, since the system and reservoir do not interact at this time:

$$\hat{\chi}(0) = \hat{\rho}_S(0) \otimes \hat{\sigma}_i(0)$$

At time $t = t_i$, the system interacts with the i-th meter according to the interaction Hamiltonian $\hat{H}_{SR}$. Since the meter is usually a laser mode it travels at the speed of light and therefore, the interaction time Δt is very small. At time $t = t_i + z_0/c$, a measurement is performed on the mode by way of projecting the combined system-reservoir state onto the measurement operator $\hat{C} = \mathbb{1}_{\mathrm{S}} \otimes \hat{C}_{\mathrm{R}}$. A partial trace over the reservoir yields the conditional system density operator $\hat{\rho}_c(t_i + z_0/c) = \mathrm{tr}_{\mathrm{R}}[\hat{C}\chi(t_i + z_0/c)]$. The subscript denotes that the density operator is conditioned on the random outcome of the measurement. The evolution path obtained by conditioning according to the measurement outcome is called a *quantum trajectory*.

Now we assume that (i) the system-reservoir interaction is time-independent, that is, $\hat{H}_{\mathrm{SR}} = \sqrt{M}(sr^{\dagger} + s^{\dagger}r)$, where M is the measurement (coupling) strength (this is justified by the assumption that the interaction takes place over an interval Δt that is small compared to the time scale for the internal dynamics of the system); (ii) the meters are non-interacting, so that $R(t) = \sum_i R_i(t)$ which denotes the string of meters (in other words, the reservoir) satisfies $[R(t), R^{\dagger}(t')] = \delta(t - t')$ since each meter is orthogonal, viz., $[R_i(t), R_j^{\dagger}(t)] = \delta_{ij}$; and (iii) the retardation time z_0/c is insignificant (since the meter for most quantum control application is laser light).

With the above assumptions, the propagator for the quantum system is:

$$\hat{U}(t_i + \Delta t, t_i) = \vec{T} \exp\left[\sqrt{M}\hat{s}\int_{t_i}^{t_i+\Delta t} \hat{R}_{\mathrm{in}}^{\dagger}(t')\,dt' - \sqrt{M}\hat{s}^{\dagger}\int_{t_i}^{t_i+\Delta t} \hat{R}_{\mathrm{in}}(t')\,dt'\right]$$

where $\vec{T}$ is the time-ordering and $\hat{R}_{\mathrm{in}}$ is a reservoir creation operator prior to the interaction, which occurs at time t_i for an interval Δt. It can be shown that the measurement result (often called the 'photocurrent' for historical reasons) is:

$$y(t) = \sqrt{M}\langle\hat{s}(t)\rangle \tag{2}$$

Thus, the measurement provides information about the system through $\langle \hat{s} \rangle$. Now, to find the state of the system (i.e., the density operator) immediately after the measurement, let us start by defining a set of coarse-grained reservoir operators as follows:

$$\hat{c}_i = \frac{1}{\sqrt{\Delta t}} \int_{t_i}^{t_i+\Delta t} \hat{R}_{\text{in}}(t)\, dt$$

In terms of these, and assuming that each meter is in its ground state, the (conditional) evolution of the combined density operator in the present Markovian approximation is given by:

$$\begin{aligned}\hat{\chi}(t_i + \Delta t) &= \hat{\rho} \otimes \hat{\sigma}_i - i\sqrt{M\Delta t}\left(\hat{s}\hat{\rho}(t) \otimes \hat{c}^\dagger \hat{\sigma}_i - \hat{\rho}(t)\hat{s}^\dagger \otimes \hat{\sigma}_i \hat{c}_i\right) \\ &\quad + M\Delta t\left(\hat{s}\hat{\rho}(t)\hat{s}^\dagger \otimes \hat{c}_i^\dagger \hat{\sigma}_i \hat{c}_i - \frac{1}{2}\hat{s}^\dagger \hat{s}\hat{\rho}(t) \otimes \hat{c}_i \hat{c}_i^\dagger \hat{\sigma}_i - \frac{1}{2}\hat{\rho}(t)\hat{s}^\dagger \hat{s} \otimes \hat{\sigma}_i \hat{c}_i \hat{c}_i^\dagger\right)\end{aligned}$$

where we notice terms of the form $\mathcal{D}[\hat{s}]\hat{\rho}(t) = \hat{s}\hat{\rho}(t)\hat{s}^\dagger - \frac{1}{2}\left(\hat{s}^\dagger \hat{s}\hat{\rho}(t) + \hat{\rho}(t)\hat{s}^\dagger \hat{s}\right)$; these are the so-called 'Linblad superoperators' describing dissipation in the Master equation for open quantum systems. Most coherent control schemes use, or propose to use, a laser field for the measurement. In that light, we assume a quadrature detection model [4]; then taking a partial trace over the reservoir and going to the limit $\Delta t \rightarrow dt$, we get the following conditional Master equation:

$$\begin{aligned}d\hat{\rho}_c(t) &= M\mathcal{D}[\hat{s}]\hat{\rho}_c(t)dt + \sqrt{M}\left(\hat{s}\hat{\rho}_c(t) + \hat{\rho}_c(t)\hat{s}^\dagger - \text{tr}[(\hat{s} + \hat{s}^\dagger)\hat{\rho}_c(t)]\hat{\rho}_c(t)\right) dW \\ &= M\mathcal{D}[\hat{s}]\hat{\rho}_c(t)dt + \sqrt{M}\mathcal{H}[\hat{s}]\hat{\rho}_c(t)dW(t)\end{aligned} \tag{3}$$

We see that the first term is a Linblad term corresponding to decoherence of the system at a rate proportional to M. The second term, called an 'Ito increment' is a Gaussian random process with zero mean and variance dt. The conditional evolution (or quantum trajectory) result for the measurement, or photocurrent, is given by

$$y(t)dt = 2\sqrt{M}\langle\hat{s}\rangle_c(t)dt + dW \quad (4)$$

The second term on the RHS thus acquires the physical meaning of the laser shot-noise. We see that when evolving the density matrix according to the equation (3) above, each time a measurement of $\langle\hat{s}\rangle$ is performed, the photocurrent is seen to be different from the value expected from equation (2) and needs to be corrected by the Ito increment – this is the so-called 'innovation' process.

Finally, from the measured photocurrent we construct a feedback signal to control $\langle\hat{s}\rangle$. This appears as a feedback Hamiltonian to modify equation (3) as follows:

$$d\hat{\rho}_c(t) = -i\left[u(t)\hat{H}_{\mathrm{Fb}}, \hat{\rho}_c(t)\right]dt + M\mathcal{D}[\hat{s}]\hat{\rho}_c(t)dt + \sqrt{M}\mathcal{H}[\hat{s}]\hat{\rho}_c(t)\,dW(t) \quad (5)$$

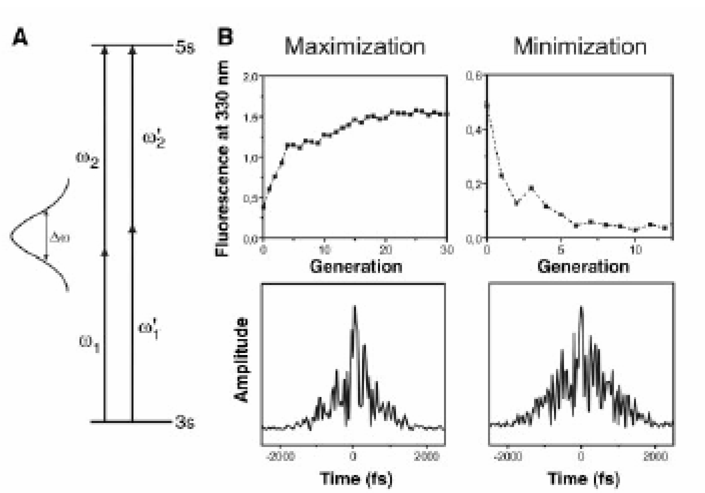
where $u(t)$, the feedback strength is calculated by convolving the photocurrent with the feedback system impulse response as shown in equation (1) earlier:

$$u(t) = \int_0^t F(t-\tau)I_c(\tau)\,d\tau \quad (6).$$

The important point of difference between equations (6) and (1) is that in this case, the input to the feedback system is random. This implies that a steady state design of the feedback block, feasible for the classical case, would not work here. In the quantum case, one would need to perform the integration in equation (6) in real-time. This is computationally expensive, but not impossible. Instead of real-time feedback, one might choose to do what is called learning control. Here, one begins with steady-state closed-loop trial designs and a learning algorithm suggests improvements of the laser field until the control objective is met. This approach has been used by Rabitz et al. in the their work, a sample of which [7] is mentioned in the next section.

IV. APPLICATIONS OF QUANTUM CONTROL

The first broad class of applications addresses the chemist's dream: to create novel stable or meta-stable products by selectively making or breaking chemical bonds to drive the chemical system toward the desired (quantum) state [7]. These applications take advantage of the enormous strides in recent times of femtosecond laser pulse-shaping capabilities. The hope is that the laser-driven coherent manipulation of the molecular motion-induced quantum interferences (between different pathways from one state to another) will facilitate products or molecular states, not accessible by conventional chemical or photo-chemical means. For example, Rabitz et al. [7] have illustrated this by considering the analog of a double-slit experiment in the 3s → 5s two-photon transition in atomic Na. Consider two (or more, as would be the case for an ultra-short laser pulse) different paths with photon energies ($\omega_1 + \omega_2$) and ($\acute{\omega}_1 + \acute{\omega}_2$); by choosing the phase difference between the path appropriately (a femtosecond laser is tailored to modulate the phases using a pulse shaper controlled by a learning algorithm), one can achieve constructive (or destructive) interference leading to maximization (minimization) of the two-photon transition probability as shown in the figure below.



In a very significant work in the quantum control of chemical systems, Herek et al. [8] have used coherent learning control over the energy flow pathways in the light-harvesting molecular complex in a photosynthetic bacterium. Their experiment on a (soft) condensed phase system not only proves that molecular complexity need not preclude the possibility of coherent control, but also opens up the new application area of the quantum control of biological systems.

That brings us to our final topic: the application of coherent control to semiconductors and their nanostructures, specifically, the carrier dynamics therein. It might be noted that, unlike the coherent control of chemical reactions, this application area has not yet witnessed a serious experimental demonstration as far as I am aware. The selected control schemes described are all theoretical proposals.

The first example we look at is the control of carrier dynamics in a bulk semiconductor within a two band effective mass model [9]. Dargys has here theorized that the inter-band carrier transfer can be controlled coherently using an optical pulse. The author obtains an equation for coherent transition between the bands. This together with a functional that minimizes the energy of the pulse yields a strategy, in the form of an Euler equation, for optimal control of the system.

The second example is a proposal for the laser-actuated coherent control of ferromagnetism in undoped, low Mn mole-fraction II(Mn)-VI dilute doped magnetic semiconductors [10]. Here, Fernandez-Rossier et al. have shown that in these paramagnetic materials, an intense *sub-bandgap* laser radiation induces coherence between the conduction and valence bands leading to an optical exchange interaction, causing the material to enter a ferromagnetic phase at low temperatures. This, if experimentally validated might be technologically important, since these material systems had never been known to show ferromagnetism under usual conditions.

The third example we might wish to consider is the control of spin-spin interaction in doped semiconductors [11]. The authors have developed a theory of bound exciton-mediated interaction between spins localized on two impurity centers. This has been applied to shallow neutral donors in III-V semiconductors. It has been shown that when the control laser energy is tuned to the bonding – anti-bonding gap of the exciton bound by the two impurities, the coupling between the spins increases and may change from ferromagnetic to anti-ferromagnetic. This method of controlling single-spin dynamics suggests application to quantum information processing.

The last example we look at is a scheme for the control of exciton dynamics in semiconductor nano-dots for quantum logic operations [12]. In this paper, the authors propose to control the dynamics of two interacting excitons of opposite polarization (control of the spin of excitons has been experimentally demonstrated elsewhere), using circularly polarized optical pulses. The quantum operation uses the lowest four states formed by these two excitons – this is sufficient for universal quantum logic since one and two-qubit operations can be used to realize any logic operation. The femto-second laser pulses are to be shaped to finish the quantum operation within the decoherence time. The authors have then theoretically demonstrated the application of the control to the complete solution of a simple quantum algorithm.

I have tried to show that the still-evolving theory of the control of quantum systems might be a highly useful tool for nanoscience. On the one hand, it provides a general purpose mathematical framework to describe the manipulated evolution of quantum systems; this obviously holds enormous promise for experimental nanotechnology. On the other hand, it might stimulate basic questions (and provide answers to) the nature of quantum mechanics, quantum information and computing, and quantum chaos.

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