Unstable Equilibria

Stable equilibria are always characterized by periodic oscillations about the equilibrium point for small perturbations, although they may not necessarily be harmonic (period independent of amplitude), or even symmetric about the point of equilibrium. Unstable equilibria lead to a wider variety of effects. Consider the following three unstable potentials with an equilibrium at x=0:

$U(x) = \quad -kx^2/2, \quad -kx^3/(2L), \qquad -kx^4/(2L^2)$

(Although there are many possible normalizations and sets of constants, this choice makes all potentials the same at x=L, a useful reference point. All have the same characteristic time.) For the first case,

$$\frac{d^2x}{dt^2} = \frac{k}{m}x = \frac{x}{\tau^2} \qquad x(t) = x_o e^{t/\tau}$$

$$x = L \qquad at \qquad t = \tau \ln(L/x_o)$$

For the remaining two cases, the differential equation is non-linear, for which there is no systematic technique for finding solutions. A systematic approach is to use the energy integral

$$\int_0^t dt = \sqrt{\frac{m}{2}} \int_{x_o}^{x(t)} \frac{dx}{\sqrt{U_o - U(x)}}$$

For the two cases, the result can be written for x(t) starting from x_o with no velocity

$$t = \tau\sqrt{L} \int_{x_o}^{x} \frac{dx}{\sqrt{x^3 - x_o^3}} \qquad t = \tau L \int_{x_o}^{x} \frac{dx}{\sqrt{x^4 - x_o^4}}$$

In terms of a dimensionless $\delta = x/x_o$,

$$\frac{t}{\tau} = \sqrt{\frac{L}{x_o}} \int_1^{\delta} \frac{d\delta}{\sqrt{\delta^3 - 1}} \qquad \frac{t}{\tau} = \frac{L}{x_o} \int_1^{\delta} \frac{d\delta}{\sqrt{\delta^4 - 1}}$$

Since we are only interested in the behavior for x>> x_o, that is $\delta >> 1$, it is very convenient to write the integrals in the form

$$\int_1^{\delta} = \int_1^{\infty} - \int_{\delta}^{\infty}$$

The first integral is a definite integral that evaluates to a constant, call them K_3 and K_4. With sufficient research, one could find them in reference books or perhaps Maple or Mathematica. They are constants of order one, actually in the range of 2 to 5, but the values are not important. The second integral can be easily approximated for $\delta >> 1$.

$$\frac{t}{\tau} = \sqrt{\frac{L}{x_o}} (K_3 - 2\delta^{-1/2}) \qquad \frac{t}{\tau} = \frac{L}{x_o} (K_4 - \delta^{-1})$$

These can be solved for x(t) as

$$x(t) = \frac{4x_o}{(K_3 - \sqrt{\frac{x_o}{L}}\frac{t}{\tau})^2} \qquad x(t) = \frac{x_o}{K_4 - \frac{x_o}{L}\frac{t}{\tau}}$$

This behavior differs strikingly from the first case. In these cases, x(t) diverges at finite time. (Of course, the divergence is illusory. The potential cannot have the form assumed for arbitrarily large x. Nonetheless, there is a second characteristic time limiting the solution, a time shortly before which x(t)=L. These critical times of divergence are

$$t_{crit} = \tau K_3 \sqrt{\frac{L}{x_o}} \qquad t_{crit} = \tau K_4 \frac{L}{x_o}$$

The instability is becoming progressively "weaker" as the exponent in U increases in the sense that the time to reach x=L is much larger for given small initial values. The scaling proceeds from log to root to linear.

These approximate solutions describe how the instability "ends", but are not correct for times much less than the critical time, for which x is not much larger than x_o.

A second approach is to try to solve the second-order differential equations directly. Solving higher-order nonlinear differential equations is a matter of judgment, guesswork, and trial and error. Especially after the preceding calculations, one is motivated to try a solution of the form

$$x(t) = \frac{A}{(B - t/\tau)^s} \qquad \tau^2 \frac{d^2x}{dt^2} = \frac{s(s+1)A}{(B - t/\tau)^{s+2}}$$

In fact, such forms are used very widely in differential equations where there is a singularity. Putting such a form into the differential equation imposes two conditions for the result to be a solution. The time dependence must be the same on both sides, which determines the value of s, and the values of the constants must agree. For the cubic U(x), this requires

$$s + 2 = 2s \qquad s = 2 \qquad 6A = \frac{3A^2}{2L} \qquad A = 4L$$

The constant B remains available to satisfy an initial condition x(0)= x_o.

$$B = 2\sqrt{\frac{L}{x_o}} \qquad t_{crit} = 2\tau\sqrt{\frac{L}{x_o}}$$

This is the same scaling as before, and would be numerically identical if K_3=2, which is approximately correct. Applying the same method to the quartic U gives

$$s = 1 \qquad A = L \qquad B = \frac{L}{x_o} \qquad t_{crit} = \tau\frac{L}{x_o}$$

Again, the same scaling appears as before, but with a slightly different numerical constant. The difference between the two approaches represents neither an error in the calculations nor in the approximations. Rather, the two solutions are answers to slightly different problems. The energy integral was taken from an initial energy from x_o with v_o=0. The solutions to the differential equation had only one free constant, set for x_o, but those solutions imply a $v_o \neq 0$. The solutions to the differential equation do have the advantage of being correct for all $t<t_{crit}$, in particular for small values of t.

Fortunately, this is sufficient to give a good characterization of the nature of the instability for the three potentials. It is not necessary to find the complete solutions for arbitrary x_o, v_o. (This is generally very difficult for nonlinear equations; one cannot use a superposition of solutions as for linear problems.)