A Review of Vector Calculus with Exercises

TOPICS

I. Introduction
II. Integrals: Line, Surface, and Volume
III. Gradient
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These notes provide a quick review and summary of the concepts of vector calculus as used in electromagnetism. They include a number of exercises, with answers, to illustrate the applications and provide familiarity with the manipulations. Since a vector is naturally a spatial and geometrical object, it is extremely useful to make sketches of the various functions and vector fields in the exercises.

The notation is conventional: Vectors are denoted by boldface (**r**,**A**), *unit vectors as* $\hat{\mathbf{x}}$, *and components either by subscript* A_X *or as a triplet* $(A_X;A_Y;A_Z)$.

I. Introduction

Since electromagnetism is inherently three-dimensional, the mathematical description is inevitably in three-space. (Relativity comes later). Although one could consider any system of coordinates, and some unusual ones are advantageous in certain problems, we shall consider only the conventional Cartesian (x,y,z), cylindrical (r,θ,z), and spherical (r,θ,φ) systems (cf. Figure attached). Since the fundamental concepts are phrased in terms of invariants, they can all be generalized to coordinate systems of arbitrary dimension and form, but it is more efficient to defer that treatment to tensor calculus, which provides a more natural and thorough formalism.

A vector is a geometrical object with magnitude and direction independent of any particular coordinate system. A representation in terms of components or unit vectors may be important for calculation and application, but is not intrinsic to the concept of vector. An equation **A** = **B** states an equality independent of coordinates and thus requires that the representation of **A** in any coordinate system be identical to that of **B** in that system. (If an object were specified in some manner such this were not true, the object would <u>not</u> be a vector.) Vectors satisfy addition: **D** = **A** + **B** + **C** = **A** + **C** + **B** = (**A** + **B**) + **C** = **A** + (**B** + **C**).

These properties are sufficient to justify the form $\mathbf{A} = |A|\hat{\mathbf{a}}$, where |A| is the invariant length and $\hat{\mathbf{a}}$ shares the invariant direction but has unit length. One can introduce another invariant, the scalar product $\mathbf{A}\cdot\mathbf{B} = |A||B|\,\hat{\mathbf{a}}\cdot\hat{\mathbf{b}}$. Here $\hat{\mathbf{a}}\cdot\hat{\mathbf{b}}$ is interpreted as the (invariant) length of $\hat{\mathbf{a}}$ in the direction $\hat{\mathbf{b}}$, or vise versa and is geometrically the cosine of the angle between $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$. The operation is commutative and distributive. Any vector can therefore be represented in Cartesian components: $\mathbf{A} = (A_X;A_Y;A_Z) = A_X\hat{\mathbf{x}} + A_Y\hat{\mathbf{y}} + A_Z\hat{\mathbf{z}}$, where $A_x = \mathbf{A}\cdot\hat{\mathbf{x}}$, etc. In fact, any set of orthogonal unit vectors $\mathbf{e}_i$, (that is $\mathbf{e}_i\cdot\mathbf{e}_j = 0$ for $i \neq j$) can provide a (unique) representation of a vector with the usual property that

$$\mathbf{A}\cdot\mathbf{B} = \sum_{i=1}^{3} A_i B_i$$

One can also introduce a "vector product" (**A** x **B**), but there are some surprising subtleties. It is normally defined geometrically as **C** = **A** x **B** with $|C| = |A||B| \sin\theta_{\mathbf{AB}}$ and direction perpendicular to the plane of **A**, **B** with the right hand rule. With varying degrees of rigor and effort, one then obtains the mnemonic using a determinant for a representation in orthogonal components:

$$\mathbf{C} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

Ironically, this form is closer to the fundamental definition. Recalling (or introducing) two functions from linear algebra,

δ_i^j = 1 if i=j; 0 otherwise (Kronecker delta)

ε^{ijk} = 1 if i,j,k an even permutation of 1,2,3; -1 if an odd permutation (e.g.2,1,3); 0 otherwise (i.e. if any index is repeated as in 1,2,2) (Permutation symbol)

the determinant expression is identical to the following definition:

$$C_i = \sum_{j=1}^{3} \sum_{k=1}^{3} \varepsilon^{ijk} A_j B_k$$

This is actually the best definition of vector product; it is easily generalized to tensors. It is relatively straightforward to show that it also has the expected properties. **C** is perpendicular to **A** and **B** because **C**·**A** = 0 and **C**·**B** = 0, simple consequnces of the properties of permutation symbol. The definition follows the right-hand rule (assuming $\mathbf{e}_i$ are right handed), and the equation for the magnitude can be established quickly from some identities below.

(The fact that the sign of **A**x**B** changes between right and left handed coordinate systems raises an interesting point, for vectors were <u>defined</u> as being invariant and independent of coordinate system. The resolution of this paradox is that the vector product is not a true vector; it is sometimes called a pseudo-vector. It behaves like a vector otherwise, but it is mathematically a tensor $T_{ij} = A_iB_j - A_jB_i$, technically an antisymmetric tensor of rank two. This detail has no effect on the discussion here, but may relieve some doubts that physics is making some subtle mathematical error which might have unforeseen consequences.)

Permutation symbols are quite useful once you are accustomed to them. An important result, of which you must convince yourself, is that

$$\sum_{k=1}^{3} \varepsilon^{ijk} \varepsilon^{mnk} = \delta_m^i \delta_n^j - \delta_n^i \delta_m^j$$

but a little consideration of the permutation symbol definition should suffice. It becomes very clear in retrospect. This result makes it easy to prove the standard identity

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\,\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\,\mathbf{C}$$

The relation for the magnitude of the vector product can be proven as

$$(\mathbf{B}\times\mathbf{C})\cdot(\mathbf{B}\times\mathbf{C}) = \varepsilon^{ijk}B_jC_k\,\varepsilon^{imn}\,B_mC_n = (\delta_m^j\,\delta_n^k - \delta_n^j\,\delta_m^k)B_jC_kB_mC_n$$

where the "tensor" convention of summing over repeated indices is understood. (All indices in this case.)

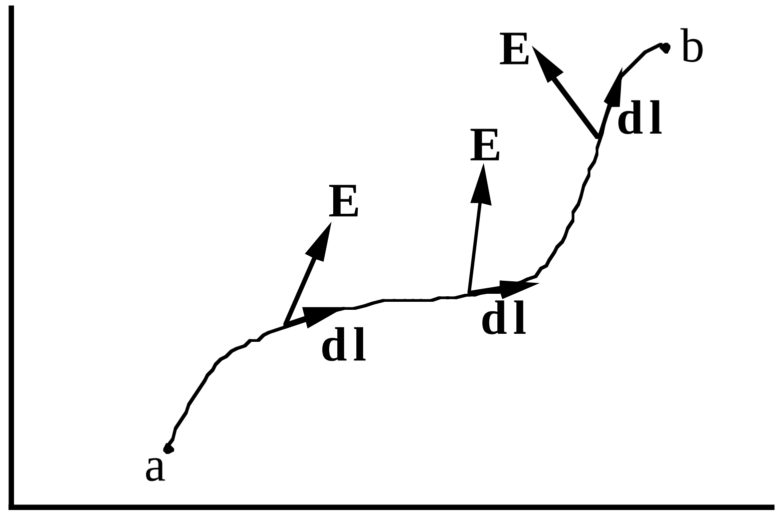
$$= |B|^2|C|^2 - (\mathbf{B}\cdot\mathbf{C})^2 = |B|^2|C|^2(1 - \cos^2\theta) = |B|^2|C|^2 \sin^2\theta$$

the required result.

II. Integrals: Line, Surface, and Volume

A common integral which arises in several physical contexts is the line integral, which is equivalent to a one-dimensional integral

$$\int_a^b \mathbf{E}\cdot d\mathbf{l} = \int_a^b |\mathbf{E}| \cos\theta \, dl$$



taken along some specified path between a and b. If one introduces a special coordinate s which measures distance along the path and assumes that all quantities can be parameterized by that coordinate, the integral becomes a simple integral

$$= \int_0^L E(s) \cos\theta(s) \, ds$$

If the path follows a coordinate line (or if one can choose a coordinate system for which the lines coincide with the path), this form is easily constructed, and the problem has probably been chosen so that the resulting integral is not too difficult. For more complex paths, however, the explicit evaluation of s, which one needs to perform the actual calculation, may be awkward. More often, the path is specified by some parameterization $\mathbf{r}(t) = (x(t); y(t); z(t))$, where t might even be one of the coordinates, and $\mathbf{E}(x,y,z)$ is also given. Then

$$d\mathbf{l} = \left(\frac{dx(t)}{dt} ; \frac{dy(t)}{dt} ; \frac{dz(t)}{dt} \right) dt$$

and one can compute either from components or magnitudes and the cosine, depending on convenience, and again arrive at a simple integral:

$$= \int_{t_1}^{t_2} f(t) \, dt$$

The procedure can be applied to other coordinate systems, but the equation for d**l** must be modified appropriately:

$$d\mathbf{l} = (dr; r\, d\theta; dz)$$

$$d\mathbf{l} = (dr; r\, d\theta; r \sin\theta \, d\varphi)$$

Exercise 1: Evaluate the following line integrals $\int \mathbf{E}\cdot d\mathbf{l}$ for:

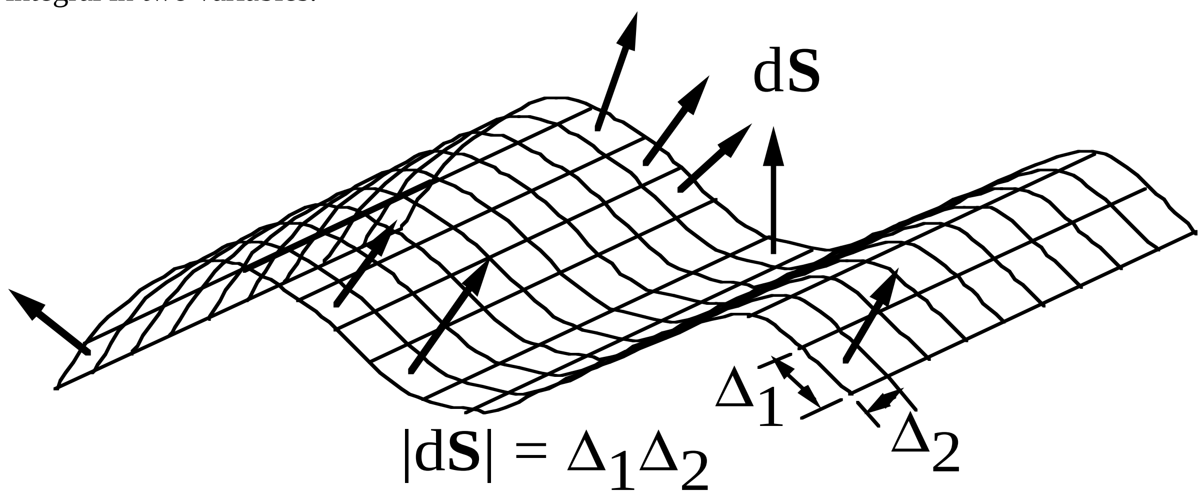
(a) $\mathbf{E} = (Ax^2y; Byz; Cxz^2)$ along the axes $(0,0,0)\to(1,0,0)\to(1,1,0)\to(1,1,1)$;

(b) $\mathbf{E} = (Ax^2y; Byz; Cxz^2)$ along the curve $(2t+1, t^2, 4t^2-1)$ from t=0 to t =1;

(c) $\mathbf{E} = (Ar\sin^2\theta; B\cos\varphi; C\sin\theta\cos\varphi)$ along spherical coordinate lines $(1,\pi/2,0)\to(1,\pi,0)\to(2,\pi,0)\to(2,\pi,3\pi/2)$;

(d) $\mathbf{E}$ = ($Ar^2 \sin\theta$; $Bz \sin\theta \cos\theta$; Cr) along cylindrical coordinate lines from (2,0,1)→(4,0,1)→(4,π,1).

Two-dimensional integrals are surface integrals, but the generalization of surface integrals in three dimensions is somewhat more complicated. The integrals of concern for physics have the form $\iint \mathbf{E}\cdot d\mathbf{S}$ over some specified surface and the vector $d\mathbf{S}$ is defined as $\hat{\mathbf{n}}\, dA = \hat{\mathbf{n}}\, dx\, dy$, for example, where $\hat{\mathbf{n}}$ is the vector normal to the surface (the only unique direction that can be associated with a surface) and some convention is specified for which direction is positive. These surface integrals are defined for all sorts of complicated surfaces. To evaluate the integrals, some mapping or parameterization of the surface in terms of two convenient variables for integration must be found. Although such general surfaces are essential for proofs and general arguments, the integrals which must generally be evaluated are much simpler. A coordinate system can usually be chosen such that the surface of integration has one of the coordinates constant (e.g. a sphere of r = a) and the other two provide natural variables on the surface. This kind of integral is easily formulated as a conventional integral in two variables.



Exercise 2: Evaluate the following surface integrals:

(a) $\mathbf{E}$ = (Axz; By; Cz^4) over a rectangle in the yz plane (1,2,2) (1,2,4)(1,3,2)(1,3,4);

(b) $\mathbf{E}$ = (Az^2y; Bxz; Cyz^2) over the simple unit cube between (0,0,0) and (1,1,1,);

(c) $\mathbf{E}$ = ($A\cos^2\varphi/r$; $B\sin^2\theta$; 0) over the surface of the sphere r = 2;

(d) $\mathbf{E}$ = (Ay; Bxz^2; Cz) [Cartesian] over the surface of the sphere r = 4;

(e) $\mathbf{E}$ = ($Ar^3 \cos\theta$; $B\sin^2\theta$; Cz^2) over the cylindrical wedge $0\le r\le 1$, $0\le\theta\le\pi/4$, $0\le z\le 2$;

(f) $\mathbf{E}$ = (Ar^2; $Br\sin\theta$; $C\cos\varphi$) over the outside conical surface $1\le r\le 2$, $\theta=\pi/3$ (this is an open surface, excluding the end faces);

Volume integrals are actually the least complicated variety of integral in three dimensions. They have the basic form

$$\iiint f(\mathbf{r})\, d^3r \quad \text{with} \quad d^3r = dx\,dy\,dz = r\,dr\,d\theta\,dz = r^2 \sin\theta\, dr\, d\theta\, d\varphi$$

for the volume element. The only possible complication is the limits of integration, the expression of the boundary of the volume. Again, one can usually choose the coordinate system so that the boundaries are easily represented, often as sections of the coordinate surfaces.

Exercise 3: Evaluate the following volume integrals:

(a) $\rho = Axyz$ for a rectangular volume with sides along coordinate planes between (1,1,2) and (3,5,6);

(b) $\rho = Ae^{-\alpha r}$ for the cylinder $2 \le z \le 5$, $r \le 4$;

(c) $\rho = Ax^2$ for the sphere $r \le 2$;

(d) $\rho = Ae^{-\alpha r} \sin^2 \theta$ for the sphere $r \le 3$.

III. Gradient

The gradient arises from the generalization of a derivative of a function $\phi(\mathbf{r})$ that depends on a position that requires several variables to specify. The derivative in any direction $\hat{l}$ is given by

$$\left.\frac{d\phi}{dl}\right|_{\hat{l}} \equiv \nabla\phi \cdot \hat{l}$$

where the directional derivative in the direction $\hat{\mathbf{l}}$ is given by the usual limit and this equation serves as the definition of gradient (grad $\phi \equiv \nabla\phi$). In particular, by choosing $\hat{\mathbf{l}}$ to be the unit vectors in any specified system of coordinates, explicit expressions for $\nabla\phi$ in that system may be obtained. If the coordinate is ξ,

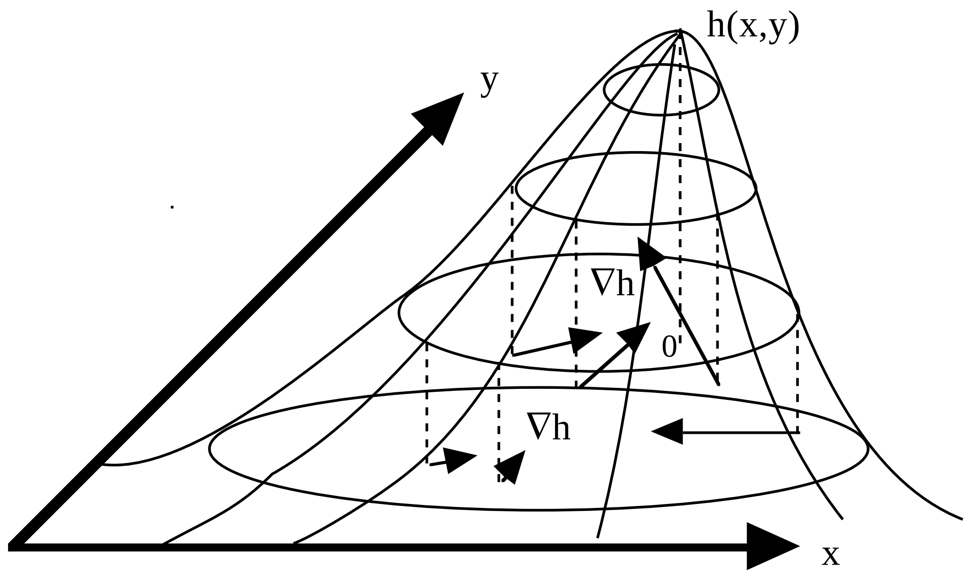
$$\nabla\phi_\xi = \left.\frac{d\phi}{dl}\right|_{\xi} = \frac{\partial\phi}{\partial\xi}\,\frac{d\xi}{dl}$$

Using this expression, verify that the expressions for gradient in the three coordinate systems are:

$$\nabla\phi = \left(\frac{\partial\phi}{\partial x}\,;\ \frac{\partial\phi}{\partial y}\,;\ \frac{\partial\phi}{\partial z}\right) \qquad \text{Cartesian}$$

$$\left(\frac{\partial\phi}{\partial r}\,;\ \frac{1}{r}\frac{\partial\phi}{\partial\theta}\,;\ \frac{\partial\phi}{\partial z}\right) \qquad \text{Cylindrical}$$

$$\left(\frac{\partial\phi}{\partial r}\,;\ \frac{1}{r}\frac{\partial\phi}{\partial\theta}\,;\ \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}\right) \qquad \text{Spherical}$$



In two dimensions, one can visualize $\phi(x,y)$ as a contour map $h(x,y)$ for which h is the altitude of the land at the point (x,y). The vertical dashed lines indicate the elevation h in the drawing. Lines of constant h are the contours of constant elevation, shown as ovals above, and the maxima and minima are peaks and holes. (One maximum shown.) The gradient vector is two-dimensional in the x-y plane for this case; ∇h is shown as an arrow for several representative points above. The direction of ∇h at each point indicates the direction of steepest rise and its magnitude is the steepness. Note that $\nabla h = 0$ at the summit (and in general at any maximum or minimum).

The principal application of gradient in electromagnetism begins with the electric potential and the electrostatic relation $\mathbf{E} = -\nabla\phi$.

<u>*Exercise 4:*</u> For the following ϕ, find $\mathbf{E} = -\nabla\phi$.

(a) $\phi = A/r$ [spherical];

(b) $\phi = A/\sqrt{x^2+y^2+z^2}$;

(c) $\phi = Axy$;

(d) $\phi = Ax + By$ {Sketch for A=B=1 and for A=1, B=4};

(e) $\phi = Ax^2 + By^2$;

(f) $\phi = A/[r(1+ \varepsilon \cos\theta)]$ {cylindrical, $\varepsilon<1$};

(g) $\phi = \frac{1}{r}\sin\theta\cos\varphi$ {spherical}.

Not all vectors can be obtained as the gradient of a scalar ϕ. The condition is that $\nabla \times \mathbf{E} = 0$, as discussed in Section IV. If $\mathbf{E} = -\nabla\phi$, ϕ can be obtained in several ways, one of which is simply to integrate the partial derivatives in sequence. For example, if $\mathbf{E} = (E_x(\mathbf{r}),E_y(\mathbf{r}),E_z(\mathbf{r}))$,

$$\phi = -\int E_x(\mathbf{r})dx + f(y,z); \qquad E_y(\mathbf{r}) = -\frac{\partial f}{\partial y} + \frac{\partial\int E_x(\mathbf{r})dx}{\partial y}$$

$$f(y,z) = \int\left(-E_y(\mathbf{r}) + \frac{\partial\int E_x(\mathbf{r})dx}{\partial y}\right)dy + g(z)$$

This expression for f(x,y) may next be substituted into $\phi = -\int E_x(\mathbf{r})dx + f(y,z)$ and the result used in $E_z(\mathbf{r}) = -\partial\phi/\partial z$ to obtain an equation for dg(z)/dz which may then be integrated to find $\phi(x,y,z)$ to within a constant. The actual process is almost easier than this description, but it is important to remember that integrating <u>partial</u> derivatives allows not <u>constants</u> of integration, but the <u>functions</u> f(y,z), g(z).

<u>Exercise 5:</u> For the following "**E**", determine if $\mathbf{E} = -\nabla\phi$, and if so, find ϕ:

(a) $\mathbf{E} = (6x-2y; -2x; 0)$;
(b) $\mathbf{E} = (3xy; 3x; 0)$;
(c) $\mathbf{E} = (6yz; 6xz; 6xy)$;
(d) $\mathbf{E} = (4x-2y; -2x-3z; -3y-2z)$;
(e) $\mathbf{E} = ((\cos^2\theta)/r^2; (2\sin\theta\cos\theta)/r^2; 0)$ [cylindrical];
(f) $\mathbf{E} = ((\cos\theta\cos\varphi)/r^3; (\sin\theta\cos\varphi)/2r^3; (\cot\theta\sin\varphi)/2r^3)$ [spherical];

Potential differences can also be determined by directly inverting the definition of directional derivative as

$$\phi(\mathbf{a}) - \phi(\mathbf{b}) = -\int_{\mathbf{b}}^{\mathbf{a}} \mathbf{E}\cdot d\mathbf{l}$$

Since the result is independent of the path chosen between **a** and **b**, one can use any path. It is often most convenient to take segments along the coordinate axes, but one could follow any curve.

<u>Exercise 6:</u> For $\mathbf{E} = (6xy; 3x^2; 0)$, find the potential of (1,1,0) with respect to the origin (0,0,0) by:

(a) Integrating the partial derivatives as in Exercise 2 to find $\phi(\mathbf{r})$ and evaluating the difference;
(b) Evaluating the line integral along the axes on the path $(0,0,0)\rightarrow(1,0,0)\rightarrow(1,1,0)$ and also along the path $(0,0,0)\rightarrow(0,1,0)\rightarrow(1,1,0)$;
(c) Evaluating the line integral along the curve $y = x^3$;
(d) Evaluating the line integral along an arbitrary y=f(x) [z=0] where one need only require that f(0) = 0 and f(1) = 1.

<u>Exercise 7:</u> For $\mathbf{E} = (2r\sin^2\theta\sin\varphi; 2r\sin\theta\cos\theta\sin\varphi; r\sin\theta\cos\varphi)$ in spherical coordinates, find the potential of $(5,\frac{\pi}{2},\frac{\pi}{2})$ with respect to (0,0,0) and (2,0,0) by:

(a) Integrating the partial derivatives to find ϕ as in Exercise 2 and evaluating the differences;
(b) Evaluating the line integral along a radial line $(r,\frac{\pi}{2},\frac{\pi}{2})$ from (0,0,0) to $(5,\frac{\pi}{2},\frac{\pi}{2})$ for the first answer {note that $(0,0,0)=(0,\theta,\varphi)$, a degeneracy};
(c) Evaluating the line integral along the radial line (r,0,0) from (2,0,0) to (5,0,0), along θ to $(5,\frac{\pi}{2},0)$, and then along φ to $(5,\frac{\pi}{2},\frac{\pi}{2})$ for the second.

IV. Divergence

The term divergence, as most of the terminology in vector calculus, derives from the application to fluid mechanics. Those origins are useful in obtaining a physical feeling for the meaning

of the mathematics. If **V**(**r**) represents the vector fluid velocity, the divergence of **V** [div **V** or $\nabla\cdot\mathbf{V}$] indicates the extent to which the flow is diverging from the point **r**; it suggests that there is a source of fluid there.
The divergence operator is defined to satisfy the divergence theorem:

$$\iint \mathbf{E}\cdot d\mathbf{S} = \iiint \nabla\cdot\mathbf{E}\ d^3r$$

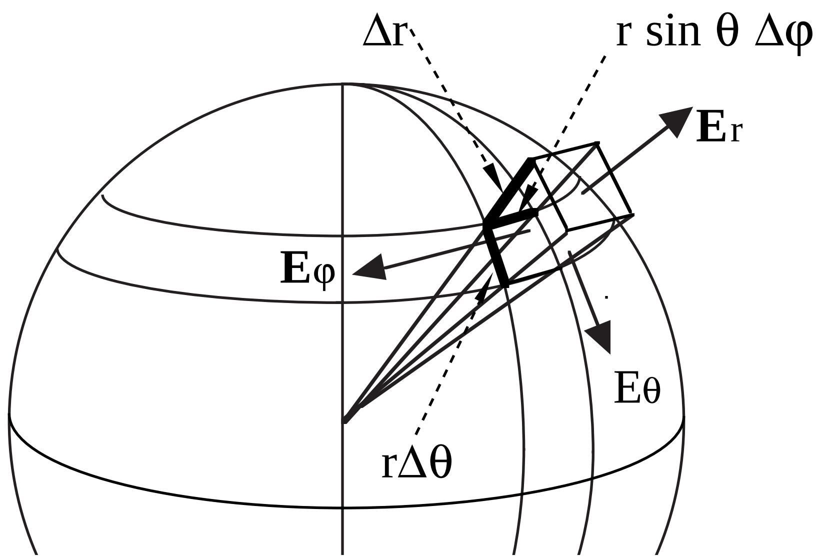
where it is understood that the surface integral covers the closed surface that encloses the volume over which the second integral is taken, and the normal d**S** points outward. By considering small volumes with sides along the coordinate axes in each coordinate system, expressions for the divergence may be obtained:

$$\nabla\cdot\mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

$$= \frac{1}{r}\frac{\partial(rE_r)}{\partial r} + \frac{1}{r}\frac{\partial E_\theta}{\partial\theta} + \frac{\partial E_z}{\partial z}$$

$$= \frac{1}{r^2}\frac{\partial(r^2E_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\ E_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial E_\varphi}{\partial\varphi}$$

For spherical coordinates, this is illustrated in the figure. The sides of the peculiar solid volume element are Δr in the radial direction and $r\Delta\theta$ in the θ direction and $r \sin\theta\ \Delta\varphi$ in the φ direction at the base. Considering in turn the pairs of r, θ, and φ faces in the surface integral, one obtains the complicated expressions $E_r(r+\Delta r)(r+\Delta r)\Delta\theta(r+\Delta r)\sin\theta\ \Delta\varphi - E_r(r)r\Delta\theta\ r\sin\theta\ \Delta\varphi + E_\theta(\theta+\Delta\theta)\Delta r\ r\sin(\theta+\Delta\theta)\ \Delta\varphi - E_\theta(\theta)\Delta r\ r\sin\theta\ \Delta\varphi + [E_\varphi(\varphi+\Delta\varphi) - E_\varphi(\varphi)]\Delta r\ r\Delta\theta$, because the lengths of the edges are along θ and φ differ for different surfaces. If one expands $f(x+\Delta x)$ in a Taylor series, this expression for the surface integral becomes



$$\left[\frac{\partial E_r}{\partial r} + 2\frac{E_r}{r}\right]\{r^2\sin\theta\ \Delta r\ \Delta\theta\ \Delta\varphi\} + \frac{1}{r}\left[\frac{\partial E_\theta}{\partial\theta} + E_\theta\frac{\cos\theta}{\sin\theta}\right]\{r^2\sin\theta\ \Delta r\ \Delta\theta\ \Delta\varphi\}$$

$$+ \frac{1}{r\sin\theta}\frac{\partial E_\varphi}{\partial\varphi}\{r^2\sin\theta\ \Delta r\ \Delta\theta\ \Delta\varphi\}$$

Since the common factor $\{r^2 \sin\theta\ \Delta r\ \Delta\theta\ \Delta\varphi\}$ is the spherical volume element, $r^2 \sin\theta\ dr\ d\theta\ d\varphi$, its coefficient should be the divergence in order to make the divergence theorem correct. One can quickly verify that the terms are indeed simply the expansion of the expression for divergence in spherical coordinates listed above.

Exercise 8: Compute the divergence of the following vector fields: (Be sure to sketch the vectors and divergence and consider the physical interpretation if the vector is fluid velocity or an electric field [charge density].)

(a) $\mathbf{E} = (3x^2; 0; 0)$;
(b) $\mathbf{E} = (6xy; 3x^2; 0)$;
(c) $\mathbf{E} = (y; x+6y+z^2; 2yz)$;
(d) $\mathbf{E} = (x; y; z)/(x^2+y^2+z^2)^{\frac{3}{2}}$;
(e) $\mathbf{E} = (r^{-2}; 0; 0)$ (spherical);
(f) $\mathbf{E} = (r; 0; 0)$ for r<b and $\mathbf{E} = (b^2/r; 0; 0)$ elsewhere [cylindrical];
(g) $\mathbf{E} = (e^{-\alpha r^2}/r^2; 0; 0)$ [spherical].

Exercise 9: Confirm the divergence theorem by evaluating both the surface and volume integrals over a unit cube with sides along the axes between (0,0,0) and (1,1,1) for the following vector fields:

(a) $\mathbf{E} = (2xyz; x^2z; x^2y)$;
(b) $\mathbf{E} = (yz; x^2z; y)$.

Exercise 10: Confirm the divergence theorem by evaluating both the surface and volume integrals over a cylinder centered on the z axis of radius b for $0 \le z \le 1$ for $\mathbf{E} = (r\cos^2\theta; r\sin^2\theta; Az)$.

Exercise 11: Confirm the divergence theorem by evaluating both the surface and volume integrals over the unit sphere centered on the origin for $\mathbf{E} = (Ar; Dr\sin\theta; 0)$. Certain aspects of this are subtle. Since the normal to the surface is only in the radial direction, only the contribution to the divergence from E_r is in some sense "counted" in the integral. Other contributions to the divergence may occur, as in this calculation, but they vanish by symmetry in the volume integral. You might think you could defeat this by choice of E_θ, e.g. $E_\theta = Dr\cos\theta$, but then the divergence diverges at $\theta=0,\pi$. You have actually introduced line charges along the z axis, exactly analogous to the case $\mathbf{E} = (r^{-2}; 0; 0)$, which appears superficially to have div $\mathbf{E}=0$ although $\iint \mathbf{E}\cdot d\mathbf{S} = 4\pi$ because the divergence is singular at r=0 and is actually a delta function point charge. The divergence theorem must be used very carefully if div $\mathbf{E}$ diverges at any point in the volume.

Exercise 12: Confirm the divergence theorem by evaluating both the surface and volume integrals over the unit northern hemisphere centered on the origin with its bottom plane defined by $\theta=\pi/2$ for $\mathbf{E} = (Ar; Dr\sin\theta; 0)$.

V. Laplace and Poisson Equations

The combination $\mathbf{E} = -\nabla\phi$ and $\nabla\cdot\mathbf{E} = \rho(\mathbf{r})/\varepsilon_o$ define the Poisson equation:

$$\nabla^2\phi = -\rho(\mathbf{r})/\varepsilon_o$$

where the Laplacian operator ∇^2 is defined as the divergence of the gradient using the operators already cited and may be summarized as:

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial\phi}{\partial r}) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial\phi}{\partial r}) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\phi}{\partial\theta}) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2}$$

Exercise 13: Determine the charge density required to support the following potentials: [Note that a particular solution for $\phi(\mathbf{r})$ depends not only on $\rho(\mathbf{r})$ but also on the boundary conditions or symmetry.]

(a) $\phi = Axyz$;

(b) $\phi = A(x+y+z)$;

(c) $\phi = Ar^2$ (both cylindrical and spherical);

(d) $\phi = Ax^2$;

(e) $\phi = Ay^2z$;

(f) $\phi = r^2 \sin 3\theta$ (cylindrical);

(g) $\phi = r^3 \cos\theta \sin^2\varphi$.

Much of electrostatics as well as a number of other physics problems reduce to solving the Laplace or Poisson equations, given $\rho(\mathbf{r})$ and boundary conditions. It is the boundary conditions which actually cause the greatest complexity.

VI. Curl and Stokes Law

The vector operator curl (or circulation) is constructed to satisfy Stokes Law as

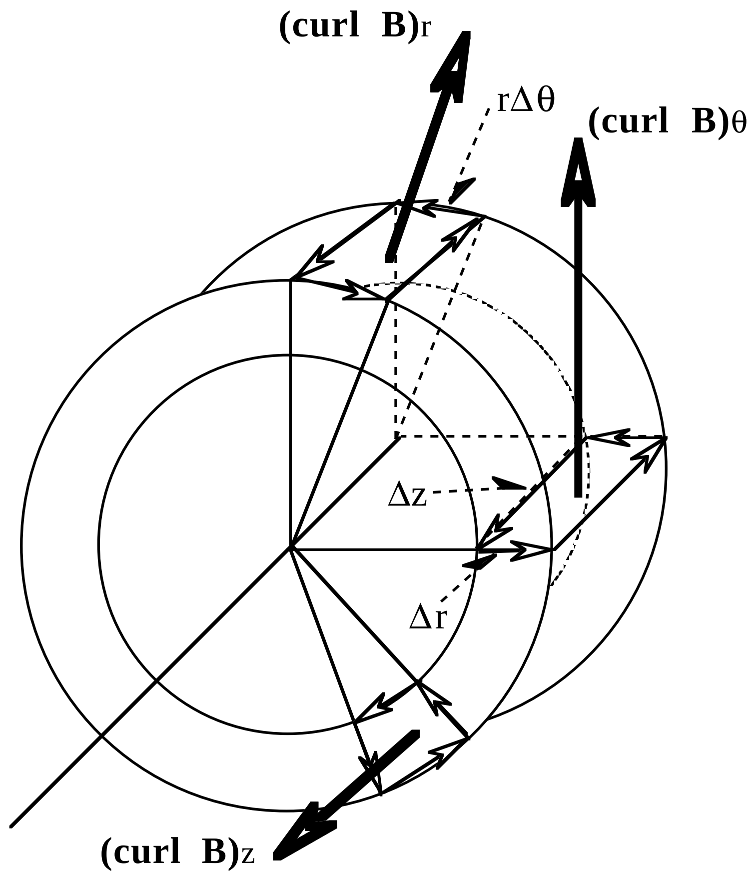
$$\oint \mathbf{B}\cdot d\mathbf{l} = \iint \nabla x\mathbf{B}\cdot d\mathbf{S}$$

where the surface S is any surface bounded by the closed curve around which the line integral is taken, and the positive normal is determine by the right hand rule following the direction of the line integral. By considering differential areas defined by increments along the coordinate axes, formulas can be obtained in various systems:

$$\nabla x\mathbf{B} = \left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}; \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}; \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)$$

$$= \left(\frac{1}{r}\frac{\partial B_z}{\partial\theta} - \frac{\partial B_\theta}{\partial z}; \frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r}; \frac{1}{r}\frac{\partial(rB_\theta)}{\partial r} - \frac{1}{r}\frac{\partial B_r}{\partial\theta}\right)$$

$$= \left(\frac{1}{r\sin\theta}\left[\frac{\partial(\sin\theta\, B_\varphi)}{\partial\theta} - \frac{\partial B_\theta}{\partial\varphi}\right]; \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial B_r}{\partial\varphi} - \frac{\partial(rB_\varphi)}{\partial r}\right]; \frac{1}{r}\left[\frac{\partial(rB_\theta)}{\partial r} - \frac{\partial B_r}{\partial\theta}\right]\right)$$



An example of this calculation for cylindrical coordinates is provided in the figure. The θ component of curl is obtained by considering the Δr, Δz loop. The $\oint \mathbf{B}\cdot d\mathbf{l} = B_z(r)\Delta z + B_r(z+\Delta z)\Delta r - B_z(r+\Delta r)\Delta z - B_r(z)\Delta r - = (-\partial B_z/\partial r + \partial Br/\partial z)\Delta r\Delta z$, which is indeed the component of curl times the element of area $\Delta r\Delta z$ as in Stokes law. The radial component is similar with $\oint \mathbf{B}\cdot d\mathbf{l} = B_\theta(z) r\Delta\theta + B_z(\theta+\Delta\theta)\Delta z - B_\theta(z+\Delta z)r\Delta\theta - B_z(\theta)\Delta z = \left(-\frac{\partial B_\theta}{\partial z} + \frac{1}{r}\frac{\partial B_z}{\partial \theta}\right) r\Delta\theta\Delta z$, again the component of curl times the element of area. The z component is the most complicated with $\oint \mathbf{B}\cdot d\mathbf{l} = B_r(\theta)\Delta r + B_\theta(r+\Delta r)(r+\Delta r)\Delta\theta - B_r(\theta+\Delta\theta)\Delta r - B_\theta(r)r\Delta\theta = \left(-\frac{1}{r}\frac{\partial B_r}{\partial\theta} + \frac{\partial B_\theta}{\partial r} + \frac{B_\theta}{r}\right) r\Delta r\Delta\theta$, which is again the expansion of the expression for curl times the element of area. Note that one must use a right-handed coordinate system and obey the sign conventions to obtain the correct result.

Exercise 14: The significance and terminology of curl can be seen by considering a fluid rotating about the z axis as a rigid body. The velocity is simply $\mathbf{v} = (0; \omega r; 0)$ in cylindrical coordinates. Confirm that $\nabla \times \mathbf{v}$ is the angular velocity $\omega\hat{\mathbf{z}}$. Verify Stokes law using a circle r=a.

An important consequence of Stokes law, which was used in Section I, is its implication for path independence of integrals.

$$\oint \mathbf{E}\cdot d\mathbf{l} = 0 \qquad \Leftrightarrow \qquad -\int_{\mathbf{b}}^{\mathbf{a}} \mathbf{E}\cdot d\mathbf{l} = \phi(\mathbf{a}) - \phi(\mathbf{b})$$

If the line integral is path-independent, the integral around a closed path must be zero, and vise versa. This may be easily seen because the difference between any two path integrals between **a** and **b** is a closed path integral. Therefore $\nabla\times\mathbf{E} = 0$ is a necessary and sufficient condition for the existence of a potential, $\mathbf{E} = -\nabla\phi$.

Actual calculations with Stokes law are primarily associated with the two curl equations of electromagnetism.

Exercise 15: For each of the following magnetic fields, compute the curl (the associated current distribution) and confirm Stokes law by computing each integral as suggested:

(a) **B** = $(Axy; -Ay^2/2; Cx)$ for the rectangle (1,1)(1,3)(2,1)(2,3) [z=0];

(b) **B** = $(0; Ar^2; 0)$ [cylindrical] for the circle r = 6 and computing the surface integral both for the plane surface and for a hemisphere bounded by the circle;

(c) **B** = $(0; 0; Ar \sin\theta)$ [spherical] for the equator r = 2 and the equatorial plane.

<u>*Exercise 16:*</u> As an example of a vector field with both divergence and curl, consider a velocity field in cylinrical coordinates $(V_o/r; V_o/\alpha; 0)$. This describes a fluid flow with streamlines $r = \alpha(\theta-\theta_o)$, which are spirals, the reverse of flow down a drain. Sketch and check this statement. Confirm that the divergence is zero, except for the delta-function source at r=0, and that the curl is $(V_o/\alpha r)\,\hat{\mathbf{z}}$.

ANSWERS

1. (a) C/3
   (b) 2.111A + .4666B + 3.877C
   (c) $\pi B/2$
   (d) 0

2. (a) 6A
   (b) C/2
   (c) $4\pi A$
   (d) $256\pi C/3$
   (e) $4\pi C/8$
   (f) $7\pi B/2$

3. (a) 840A
   (b) $6\pi A$
   (c) $128\pi A/15$
   (d) $(8\pi A/3)[2/\alpha^3 - (9/\alpha + 6/\alpha^2 + 2/\alpha^3)e^{-3\alpha}]$

4. (a) $\phi = A/r$ (spherical);
   (b) $\phi = A/\sqrt{x^2+y^2+z^2}$;
   (c) $\phi = Axy$;
   (d) $\phi = Ax + By$
   (e) $\phi = Ax^2 + By^2$;
   (f) $\phi = A/[r(1+ \varepsilon \cos\theta)]$ {cylindrical, $\varepsilon<1$};
   (g) $\phi = \frac{1}{r} \sin\theta \cos\varphi$ {spherical}.

5. (a) $\phi = -3x^2+2xy$;
   (b) None;
   (c) $\phi = -6xyz$;
   (d) $\phi = -2x^2 + 2xy + 3yz + z^2$;
   (e) $\phi = (\cos^2\theta)/r$;
   (f) $\phi = (\cos\theta \cos\varphi)/2r^2$;

6. $\phi = -3x^2y$; $\Delta\phi = -3$.

7. $\phi = -r^2 \sin^2\theta \sin\varphi$; $\Delta\phi = -25$.

8. (a) $6x$
(b) $6y$
(c) $6 + 2y$
(d) $4\pi\delta(x)\delta(y)\delta(z)$
(e) $4\pi\delta(r)/r^2$
(f) 2 for $r<b$; 0 for $r\geq b$
(g) $(-2\alpha e^{-ar^2})/r + 4\pi\delta(r)/r^2$

9. (a) 1/2
(b) 0

10. $(A+1)\pi b^2$

11. $4\pi A$

12. $2\pi A + 2\pi D/3$

13. (a) 0
(b) 0
(c) $-4\varepsilon_o A$, $-6\varepsilon_o A$
(d) $-2\varepsilon_o A$
(e) $-2\varepsilon_o zA$
(f) $5\varepsilon_o \sin 3\theta$
(g) $-10\varepsilon_o r \cos\theta \sin^2\varphi + 2\varepsilon_o r(\cos\theta/\sin^2\theta)(\sin^2\varphi - \cos^2\varphi)$

15. (a) $\mu_o \mathbf{j} = (0; -C; -Ax)$; $-3A$
(b) $\mu_o \mathbf{j} = (0; 0; 3Ar)$; $432\pi A$
(c) $\mu_o \mathbf{j} = (2A\cos\theta; -2A\sin\theta; 0)$; $8\pi A$