SUPERSYMMETRIC $SU(5) \times U(1)$ MODEL AND QUARK YUKAWA COUPLINGS

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Abstract

The Standard Model (SM) of elementary particles, in spite of its great success, does not address some fundamental questions, namely the different couplings for the subatomic interactions, the amount of parameters in the model, the quantization of the charge and the hierarchy problem. Glashow and Georgi proposed a model in which the strong, weak and electromagnetic interactions unify under only one gauge group: $SU(5)$. The minimal $SU(5)$ model is not consistent within the gauge couplings evolution nor with experimental data about proton decay. The supersymmetric extension of $SU(5)$ solves these problems and yields a more consistent model. In this work we study the general features of the supersymmetric $SU(5)$ model and analise the fermion mass hierarchy problem. In oder to tackle this problem, we implement a mechanism *á la* Froggatt-Nielsen assuming that at the GUT scale a horizontal $U(1)_H$ global flavour symmetry is unbroken. As a result of breaking the horizontal symmetry through an $SU(5)$ adjoint, we obtain the Yukawa couplings for the down-type quarks and some couplings for the up-type quarks as well.

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Introduction

"In science one tries to tell people,
in such a way as to be understood by everyone,
something that no one ever knew before.
But in poetry, it's the exact opposite."
Paul Dirac

According to the registered history, the first inquires of humankind about Nature were in the 7th century B.C. Thales was the first who tried to give an explanation about something called $\varphi\acute{\upsilon}\sigma\iota\varsigma$ (*physis*). According to his approach, the fundamental and more primary substance of Nature was water. We believe this was the first attempt to get a theory of everything. A set of theories followed the one introduce by Thales: Anaximenes proposed the air as the fundamental component of everything and Anaximander introduced the *apeiron* which was the infinite, eternal and indefinite. Democritus and Leucipus affirmed that all matter was composed by indivisible particles called $\alpha\tau o\mu o\iota$(´atomoi). In some sense, these particles can be regarded as the first elementary particles. The rest of such developments goes through a continuous line that includes Descartes, Newton, Dalton, Rutherford, Bohr, Einstein, Dirac, Weinberg, Salam and Glashow among others. It is clear the search of a unified theory, within all known phenomena can be explained, is not really a new subject.

In modern physics, the search of an unified description of the fundamental interactions has motivated a great amount of research and very sophisticated theoretical developments. The first brilliant unification was achieved by Maxwell putting together both the electromagnetic and optical phenomena under the same formalism. Some decades later, Einstein gave us the generalisation of all physics laws to all inertial systems in his Special Relativity Theory and, afterward, the unification of gravity and geometry with the General Relativity. Another physicist that contributed to strengthen the looking for an unification was Paul Dirac, who proposed a method: the symmetry in physics. This way let the particle physics formulate the theory that became the Standard Model of elementary particles which has been the most successful theory of fundamental physics.

There have been several attempts to get a theory of everything. Even in the case that one of the candidates works, we believe there will be always something new, unexpected and exciting. Maybe in many years we find the quarks are no elementary any more and our universe is very different to the universe we know at this moment. That is the gracefulness of Physics, that makes it one of the most precious activities of the human mind and that is a reason to work on it.

In this work we present a description of the Grand Unified Theory based in gauge group $SU(5)$. This was one of the pioneers in this type of theories. When we look for physics beyond the SM we find a set of unaddressed problems such as different gauge couplings, a large number of parameters of the model, non theoretical explanation for the charge quantisation and the fermion mass hierarchy. Specifically, the last problem refers to the large difference of mass among fermion generations. Georgi and Glashow proposed in 1974 the $SU(5)$ model to solve some of the problems mentioned above [Georgi and Glashow 1974]. The model places a fermion generation in only two $SU(5)$ irreducible representations. The boson content is enlarged by the presence of heavy bosons mediating process between quarks and leptons, these are named *leptoquarks*. This original model has been already ruled out by experimental decay data. Furthermore, there is no really unification of the gauge coupling constants. The supersymmetric extension of the model is still consistent with both experimental proton decay as well as the gauge coupling unification. This is treated in the first chapter.

However, supersymmetric (SUSY) $SU(5)$ does not solve the fermion mass hierarchy problem. Froggatt and Nielsen have given a possible solution to the problem by introducing an Abelian flavour horizontal symmetry $U(1)_H$ unbroken at the GUT scale[Froggatt and Nielsen 1979]. When this symmetry is broken through a $SU(5)$ singlet S, there arises an effective operator that generates a suppression term $\langle S\rangle/M_H$. Where M_H is the heavy Froggat-Nielsen field mass. The set of effective operators yields to the hierarchy mass structure for fermions in the SM. The description of this model is presented in the second chapter.

In the third chapter we study a $SU(5) \times U(1)_H$ model in order to find the coefficients that appear as consequence of the $U(1)_H$ breaking. Here, we use the Froggatt-Nielsen mechanism with a difference: we use an adjoint representation to break the horizontal symmetry. With this mechanism, we reproduce the mass hierarchy and find the explicit Yukawa coupling for charged leptons and down-type quarks. Also, we find a $b - \tau$ mass unification which fits with expected data.

There has been a previous work in these topics by professor Enrico Nardi, Diego Aristizábal and Fernando Duque [Aristizabal and Nardi 2004], [Aristizabal 2003], [Duque 2006].

Chapter 1
$SU(5)$ Grand Unified Theory

“Our hypotheses may be wrong and our speculations idle,
but the uniqueness and simplicity of our scheme
are reasons enough that it be taken seriously.”
H. Georgi, S. Glashow

1.1 Puzzles in the Standard Model

The SM is a gauge theory based on the gauge symmetry group $SU(3)_C \times SU(2)_L \times U(1)_Y$. This theory describes the strong, weak and electromagnetic interactions through the exchange of the following gauge fields: 8 massless gluons, 3 massive bosons ($W^{\pm}$ and Z), and a massless photon. The main ideas which yield to this model were formulated by Glashow, Weinberg and Salam in [Glashow 1961], [Weinberg 1967], [Salam 1968].

The SM contains three generations of fermion fields, where a generation has the structure:

$$L = \begin{pmatrix} \nu_l \\ l \end{pmatrix}_L, \; Q = \begin{pmatrix} u \\ d \end{pmatrix}_L, \; l_R, \, u_R, \, d_R \tag{1.1}$$

Gauge symmetry is broken through the Spontaneous Symmetry Breaking mechanism (SSB), namely:

$$SU(3)_C \times SU(2)_L \times U(1)_Y \rightarrow SU(3)_C \times U(1)_{QED} \tag{1.2}$$

Gauge bosons acquire mass through the SSB mechanism induced by the vacuum expected value (VEV) of the Higgs doublet. [Pich 1994].

The SM has successfully described almost all known experimental facts in particle physics. However, none theory in physics is complete and the SM is not the exception. There are some puzzles or problems that are beyond the SM [Mohapatra 2002].

Gauge problem

In the SM three gauge groups are associated to three different gauge couplings constants. This theory can be characterised by five parameters: α_{EM}, α_s, $\sin^2\theta_W$, m_h and M_Z. In some frameworks the gauge interactions are unified in a single gauge group. Supersymmetric (SUSY) $SU(5)$ and $SO(10)$ are two examples of models in which this is accomplished. Other theory that is concerned with this is Superstring theory.

Higgs problem

The Higgs boson existence has not been proved experimentally. In addition, there is a sensitivity in the corrections to Higgs mass when we consider physics beyond the SM. In the loop correction to the mass, there can be a quadratic divergence that implies the bare mass to be at Planck order, but the measurable mass is at the Weak scale. There is no a theoretical explanation for this fact.

Massless neutrinos

The SM does not contain right handed neutrinos ant there is no masses for neutrinos. However, using the SM fields, it is possible to write an $SU(2) \times U(1)$ invariant operator wich can lead to an effective mass for neutrino after the SSB; namely,

$$\mathcal{L}_\nu = \frac{\lambda}{M}\psi_L^T C^{-1}\tau_2\tau\psi_L \cdot \phi^T\tau_2\tau\phi \rightarrow \qquad m_{\nu_e} \simeq \frac{4m_W^2}{g^2 M}\lambda. \tag{1.3}$$

C is given in equation 1.4, τ refers to Pauli's matrices and ϕ is the Higgs field. The experimental value for electron neutrino masses is $m_{\nu_e} < 2.4$ eV. This implies a new scale M above 10^6 GeV.

The neutrino mass problem can be addressed within the see-saw mechanism.

Gravity problem

Einstein theory of gravity is not a of gauge theory. A gauge theory of gravity is not renormalizable and it does not seem to follow the local gauge invariance principle. The gauge theory of gravitation is being worked in the frameworks of Kaluza-Klein theories, Superstring theory, Loop Quantum Gravity and others.

Quantization of electric charge

There is no a fundamental explanation for the charge quantization. In $SU(5)$ the charge quantization is the result of the representation choice for the fermions.

Fermion problem

In the SM there is no theoretical explanation for the mass hierarchy among the different generations. The ratio between the charm and top masses is $m_c/m_t \approx$

0.0037 and between the up and charm masses is: $m_u/m_c \approx 0.0034$. If the quarks of the different families have the same quantum gauge numbers and the masses are generated under the same SSB, it would be natural to find a degeneration in the masses.

1.2 The $SU(5)$ Model

In 1974, Georgi and Glashow published a paper entitled *Unity of all elementary-particle forces* [Georgi and Glashow 1974]. In this paper, a series of hypotheses were argued leading to the conclusion that a gauge theory based on the $SU(5)$ gauge group could give a unified description of the strong and electroweak interactions. In what follows, we will describe the model as well as the SUSY extension.

1.2.1 The group choice and representations

The unification gauge group (G) must contain the SM gauge group, namely $G \supset SU(3)_C \times SU(2)_L \times U(1)_Y$. Due to the SM particle content, G should admit complex representation. The larger group that follows $SU(3)$ is $SU(4)$, but this is a real group. The candidate is, therefore, $SU(5)$. Since this is an enlarged group, there will be extra gauge bosons.

We will take into account that a right-handed field can always be expressed as a left-handed field. In order to implement this, we use $\psi^c \equiv C\bar{\psi}^T$ which contains the information of a right field. C is given by

$$C = i\gamma^2\gamma^0 = -C^{-1} = -C^\dagger = -C^T. \tag{1.4}$$

Then,

$$\psi_L^c = \frac{1}{2}(1-\gamma_5)\psi^c = C\frac{1}{2}(1-\gamma_5)\bar{\psi}^T = C[\bar{\psi}\frac{1}{2}(1-\gamma_5)]^T = C(\bar{\psi}_R)^T = (\psi_R)^c. \tag{1.5}$$

.

Th SM fermions have the following gauge quantum numbers:

$$\begin{aligned} u_L, d_L &: (3,2)_{Y=1/3} \\ d_L^c &: (3^*,1)_{Y=2/3} \\ u_L^c &: (3^*,1)_{Y=-4/3} \\ \nu_L, e_L &: (1,2)_{Y=-1} \\ e_L^c &: (1,1)_{Y=2} \end{aligned} \tag{1.6}$$

Where, in $(m,n)_Y$, m is the $SU(3)$ quantum number, n is the $SU(2)$ one and the subindex is the hypercharge.

Dim	$(SU(3), S(2))_Y$
5	$(3,1)_{-2/3} \oplus (1,2)_1$
10	$(3,2)_{1/3} \oplus (3^*,1)_{-4/3} \oplus (1,1)_2$
15	$(6,1)_{-4/3} \oplus (3,2)_{1/2} \oplus (1,3)_2$
$24 = 24^*$	$(8,1)_0 \oplus (3,2)_{-5/3} \oplus (3^*,2)_{5/3} \oplus (1,3)_0 \oplus (1,1)_0$
45	$(8,2)_1 \oplus (6^*,1)_{-2/3} \oplus (3,3)_{-2/3} \oplus (3^*,2)_{-7/3}$ $\oplus(3,1)_{-2/3} \oplus (3^*,1)_{8/3} \oplus (1,2)_1$
50	$(8,2)_1 \oplus (6,1)_{8/3} \oplus (6^*,3)_{-2/3} \oplus (3^*,2)_{-7/3}$ $\oplus(3,1)_{-2/3} \oplus (1,1)_{-4}$

Table 1.1: Standard Model Quantum numbers

The $SU(5)$ representations must include these representations. The possible irreducible representations (irrep) for $SU(5)$ and their decompositions in SM group are presented in figure and table 1.1 [Quigg 1983].

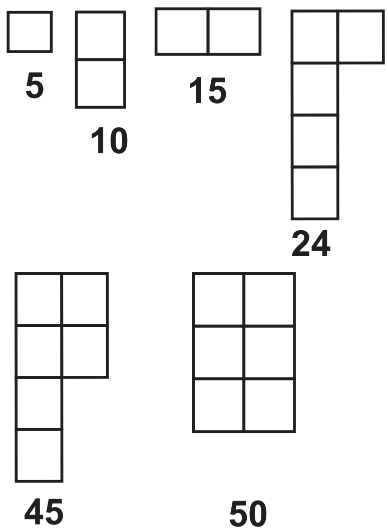


Figure 1.1: $SU(5)$ representations. Some details about these diagrams are treated in the appendix.

In table 1.1, it can be noted that the 15 as well as the 45 dimensional representation contain sextets; therefore, they are not acceptable.

According to the table, it is possible to assign to the first generation fermions the $\bar{5}$ and 10 irrep.

$$\bar{5} : \psi_j = \begin{pmatrix} d_1^c \\ d_2^c \\ d_3^c \\ e \\ -\nu_e \end{pmatrix}, \tag{1.7}$$

$$10 : \psi^{jk} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & u_3^c & -u_2^c & -u_1 & -d_1 \\ -u_3^c & 0 & u_1^c & -u_2 & -d_2 \\ u_2^c & -u_1^c & 0 & -u_3 & -d_2 \\ u_1 & u_2 & u_3 & 0 & -e^c \\ d_1 & d_2 & d_3 & e^c & 0 \end{pmatrix} \tag{1.8}$$

In fact, there is no rule for using this assignation of particles but we can see that these representations have the right quantum numbers as follows:

$$\begin{aligned} \text{For } \bar{5} &: d^c_{(3^*,1)_{2/3}} \oplus (e, -\nu_e)_{(1,2^*)_{-1}} \\ \text{For } 10 &: (u,d)_{(3,2)_{1/3}} \oplus u^c_{(3^*,1)_{-4/3}} \oplus e^c_{(1,1)_2} \end{aligned} \tag{1.9}$$

The next step is to check the gauge boson content. There are 24 bosons corresponding to 24 elements of the $SU(5)$ adjoint. The SM gauge bosons have the following numbers:

$$\begin{aligned} (8,1)_0 &\leftrightarrow \text{Gluons}, \\ (1,3)_0 &\leftrightarrow W^+_\mu, W^-_\mu, W^3_\mu, \\ (1,1)_0 &\leftrightarrow A_\mu \end{aligned} \tag{1.10}$$

$SU(5)$ contains the SM gauge bosons and extra gauge fields which are known as *leptoquarks* since they induce interactions between quarks and leptons. The quantum numbers of these new leptoquark gauge bosons are given by

$$\begin{aligned} (3,2)_{-5/3} &\leftrightarrow X_\mu, Y_\mu, \\ (3^*,2)_{5/3} &\leftrightarrow \bar{X}_\mu, \bar{Y}_\mu. \end{aligned} \tag{1.11}$$

According to the decomposition of the $SU(5)$ adjoint, we can write the boson content as follows:

$$V = \frac{1}{\sqrt{2}} \begin{pmatrix} & & & X_1 & Y_1 \\ & Gluons & & X_2 & Y_2 \\ & & & X_3 & Y_3 \\ \bar{X}_1 & \bar{X}_2 & \bar{X}_3 & \frac{W_3}{\sqrt{2}} & W^+ \\ \bar{Y}_1 & \bar{Y}_2 & \bar{Y}_3 & W^- & -\frac{W_3}{\sqrt{2}} \end{pmatrix} + \frac{1}{\sqrt{30}} B \begin{pmatrix} -2 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix} \tag{1.12}$$

This decomposition corresponds to a group of generators. The generators have been chosen such that the first eight ones are the SU(3) generators, namely L_a with $a = 1, 2, ..., 8$. The three following generators belong to SU(2): L_{8+i}, where $i = 1, 2, 3$. L_{12} is assigned to hypercharge. Then, the first twelve generators are placed in these matrices:

$$L_a = \begin{pmatrix} \lambda_a & 0 \\ 0 & 0 \end{pmatrix} \tag{1.13}$$

$$L_{8+i} = \begin{pmatrix} 0 & 0 \\ 0 & \tau_i \end{pmatrix} \tag{1.14}$$

$$L_{12} = \begin{pmatrix} C_3 I_{3\times3} & 0 \\ 0 & C_2 I_{2\times2} \end{pmatrix} \tag{1.15}$$

C_2 and C_3 are two constants that will be determined below.

The $SU(N)$ generators are normalized according to [Cheng and Li 1984]:

$$\mathrm{Tr}(L_\alpha L_\beta) = 2\delta_{\alpha\beta} \tag{1.16}$$

$$\mathrm{Tr}(L_\alpha) = 0. \tag{1.17}$$

For the $SU(3)_C$ and $SU(2)_L$ generators these conditions are automatically satisfied. For L_{12}, we must determinate the constants according to the last conditions. From (1.17), $C_3 = -\frac{2}{3}C_2$ and from (1.16), $3C_3^2 + 2C_2^2 = 2$, then it is possible to find $C_2 = \sqrt{3/5}$ and $C_3 = -\sqrt{4/15}$. So, L_{12} is

$$L_{12} = \begin{pmatrix} -\frac{2}{\sqrt{15}} I_{3\times3} & 0 \\ 0 & \frac{3}{\sqrt{15}} I_{2\times2} \end{pmatrix} \tag{1.18}$$

The hypercharge generator is proportional to generator L_{12}, $Y = CL_{12}$, and electric charge is defined as $Q = L_{11} + \frac{Y}{2} = L_{11} + \frac{1}{2}CL_{12}$.

For the $\bar{5}$ irrep, $Q = (1/3, 1/3, 1/3, -1, 0)$ or $Q(\psi_i) = \delta_{ij}Q_j$. This implies

$$\mathrm{Tr}(Q^2) = \frac{4}{3} = \mathrm{Tr}(L_{11}^2) + \frac{C^2}{4}\mathrm{Tr}(L_{12}^2) = \frac{1}{2}(1 + C^2) \tag{1.19}$$

Then,

$$C^2 = \frac{5}{3} \tag{1.20}$$

With this value, hypercharge takes a definite and non arbitrary form

$$Y = \begin{pmatrix} -\frac{2}{3} & 0 & 0 & 0 & 0 \\ 0 & -\frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & -\frac{2}{3} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.21}$$

As ψ^{jk} transforms as a tensor $(2, 0)$, it has the same quantum numbers as $\psi^j\psi^k$. Thus, the charge for **10** and **24** multiplets can be expressed as

$$
\begin{aligned}
Q(\psi^{ij}) &= Q^i + Q^j \\
Q(\psi^i_j) &= Q^i - Q^j
\end{aligned} \tag{1.22}
$$

The gauge bosons are placed in a **24** dimensional representation, as follows

$$24 = (8,1) \oplus (1,3) \oplus (1,1) \oplus (3,2) \oplus (3^*,2). \tag{1.23}$$

The three first terms correspond to the SM gauge bosons which are 12. The generators for the remaining 12 gauge boson field can be written in terms of the following matrices:

$$
\begin{aligned}
A_1 &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} \\
B_1 &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, B_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, B_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix},
\end{aligned} \tag{1.24}
$$

and the resulting generators become

$$
\begin{aligned}
L_{11+2k} &= \begin{pmatrix} 0_3 & A_k \\ A_k^T & 0_2 \end{pmatrix} \\
L_{12+2k} &= \begin{pmatrix} 0_3 & -iA_k \\ iA_k^T & 0_2 \end{pmatrix} \\
L_{17+2k} &= \begin{pmatrix} 0_3 & B_k \\ B_k^T & 0_2 \end{pmatrix} \\
L_{18+2k} &= \begin{pmatrix} 0_3 & -iB_k \\ iB_k^T & 0_2 \end{pmatrix} \\
k &= 1, 2, 3.
\end{aligned} \tag{1.25}
$$

Now, with this definition, the whole gauge boson set can be expressed as a linear combination of generator matrices:

$$A_\mu = \frac{1}{2}\sum_{a=1}^{24} A^a_\mu L_a \tag{1.26}$$

And the new gauge bosons are defined as

$$
\begin{aligned}
\overline{X}^1_\mu &= \frac{A^{13}_\mu - iA^{14}_\mu}{\sqrt{2}}, \quad \overline{X}^2_\mu = \frac{A^{15}_\mu - iA^{16}_\mu}{\sqrt{2}}, \quad \overline{X}^3_\mu = \frac{A^{17}_\mu - iA^{18}_\mu}{\sqrt{2}} \\
\overline{Y}^1_\mu &= \frac{A^{19}_\mu - iA^{20}_\mu}{\sqrt{2}}, \quad \overline{Y}^2_\mu = \frac{A^{21}_\mu - iA^{22}_\mu}{\sqrt{2}}, \quad \overline{Y}^1_\mu = \frac{A^{23}_\mu - iA^{24}_\mu}{\sqrt{2}}
\end{aligned} \tag{1.27}
$$

Finally, as a consistency check, it must be proved that model is anomaly free as it contains the SM which we already know is anomaly-free [Mohapatra 2002] [Cheng and Li 1984]. The anomaly of any fermion representation is proportional to the trace.

$$\mathrm{Tr}(T^a(R), T^b(R)T^c(R)) = \frac{1}{2}A(R)d^{abc} \tag{1.28}$$

with d^{abc} totally symmetric and T^a are the group generators in such a representation.

Since expression (1.28) is generator independent, we can select $T^a = T^b = T^c = Q$. Then,

$$\begin{aligned} \frac{A(5^*)}{A(10)} &= \frac{\mathrm{Tr}(Q^3(\psi^i))}{\mathrm{Tr}(Q^3(\psi_{ij}))} \\ &= \frac{3(1/3)^3 + (-1)^3}{3(-2/3)^3 + 3(2/3)^3 + 3(-1/3)^3 + 1} = -1 \end{aligned} \tag{1.29}$$

therefore

$$A(5^*) + A(10) = 0 \tag{1.30}$$

Thus the fermion representation that we have chosen is anomaly free.

1.2.2 Spontaneous symmetry breaking

When $SU(5)$ is unbroken, all the fields are massless. The $SU(5)$ fields acquire mass through the spontaneous breaking of the GUT gauge symmetry. The breaking occurs at a scale larger than M_W scale in order to preserve the proton stability. Thus, $M_X \gg M_W$. The phenomenon that provides masses for gauge bosons is known as the Higgs mechanism. In this mechanism scalar fields develop a vacuum expectation value (VEV). A standard description of this procedure can be found in [Cheng and Li 1984] and [Li 1964].

The SSB in the $SU(5)$ model occurs in two steps, namely

$$SU(5) \xrightarrow{\Sigma} SU(3)_C \times SU(2)_L \times U(1)_Y \xrightarrow{\Phi} SU(3)_C \times U(1)_Q \tag{1.31}$$

The first step is induced by the VEV of a Higgs multiplet placed in the adjoint 24 dimensional representation Σ. $SU(5)$ is broken at a GUT scale around 10^{16} GeV. At this scale the SM bosons are still massless. The most general scalar potential describing the interactions of Σ is

$$V(\Sigma) = -m_1^2(\mathrm{Tr}\,\Sigma^2) + \lambda_1(\mathrm{Tr}\,\Sigma^2)^2 + \lambda_2(\mathrm{Tr}\,\Sigma^4) \tag{1.32}$$

The cubic term has been omitted to leave the potential invariant under $\Sigma \to -\Sigma$. V(Σ) has an extremum at $\Sigma = \langle\Sigma\rangle$ with

$$\langle\Sigma\rangle = V \begin{pmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & -3 & \\ & & & & -3 \end{pmatrix} \tag{1.33}$$

where

$$V^2 = \frac{m_1^2}{(60\lambda_1 + 14\lambda_2)} \tag{1.34}$$

A normalisation factor $1/\sqrt{60}$ is absorbed in V.

The field is shifted to define the new set of scalar fields:

$$\Sigma' = \Sigma - \langle\Sigma\rangle = \begin{pmatrix} [\Sigma_8]^\alpha_\beta - \frac{2}{\sqrt{30}}\Sigma_0 & & & \begin{matrix}\Sigma_{X_1} & \Sigma_{Y_1} \\ \Sigma_{X_2} & \Sigma_{Y_2} \\ \Sigma_{X_3} & \Sigma_{Y_3}\end{matrix} \\ \begin{matrix}\Sigma_{\bar{X}_1} & \Sigma_{\bar{X}_2} & \Sigma_{\bar{X}_3} \\ \Sigma_{\bar{Y}_1} & \Sigma_{\bar{Y}_2} & \Sigma_{\bar{Y}_3}\end{matrix} & & & [\Sigma_3]^r_s - \frac{3}{\sqrt{30}}\Sigma_0 \end{pmatrix} \tag{1.35}$$

The mass spectrum of this scalars can be found by expressing Σ as a diagonalized matrix $\Sigma_i\delta^i_j$ and evaluating

$$\frac{\partial^2 V}{\partial\Sigma_i\partial\Sigma_l}_{\langle\Sigma\rangle} \tag{1.36}$$

Since Σ is in an adjoint representation, the covariant derivative can be written

$$\mathrm{D}_\mu\Sigma = \partial_\mu\Sigma + ig_5[A_\mu, \Sigma] \tag{1.37}$$

where g_5 is the $SU(5)$ coupling constant.

According to the redefinition of the Σ' field,

$$\begin{aligned} \mathrm{D}_\mu\Sigma &= \partial_\mu\Sigma + ig_5[A_\mu, \Sigma' + \langle\Sigma\rangle] \\ &= \mathrm{D}_\mu\Sigma' + ig_5[A_\mu, \langle\Sigma\rangle] \end{aligned} \tag{1.38}$$

Thus,

$$|\mathrm{D}_\mu\Sigma|^2 \supset g_5^2|[A_\mu, \langle\Sigma\rangle]|^2 \tag{1.39}$$

$$[A_\mu, \langle\Sigma\rangle] = \frac{V}{\sqrt{2}} \begin{pmatrix} & -5\begin{pmatrix} X_1 & Y_1 \\ X_2 & Y_2 \\ X_3 & Y_3 \end{pmatrix} \\ 5\begin{pmatrix} \bar{X}_1 & \bar{X}_2 & \bar{X}_3 \\ \bar{Y}_1 & \bar{Y}_2 & \bar{Y}_3 \end{pmatrix} & \end{pmatrix} \tag{1.40}$$

The X and Y gauge bosons masses are determined by

$$g_5^2|[A_\mu, \langle\Sigma\rangle]|^2 \sim g_5^2\frac{25}{2}V \tag{1.41}$$

such that $M_X = M_Y = \sqrt{25/2}g_5V$.

The SM gauge symmetry is spontaneously broken by the VEV of a scalar in a $5 - dim$ representation. The scalar potential in this case is given by

$$V_{\text{eff}}(\phi) = -m^2\phi^\dagger\phi + \lambda_3(\phi^\dagger\phi)^2 \tag{1.42}$$

and the VEV by

$$\langle\phi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\0\\0\\v\end{pmatrix} \tag{1.43}$$

with $v \simeq 246.2$ GeV. The $W^{\pm}$ and Z^0 gauge bosons acquire masses in a lower scale; a representative fact is that $V \geq 10^{12}v$.

$SU(5)$ has a problem that is known as *doublet triplet splitting*. Each Higgs field multiplet contains a doublet corresponding to the electroweak Higgs field, and a coloured triplet that acquires mass of the order of the GUT scale to avoid a short proton lifetime. In minimal $SU(5)$ this problem is solved using a *fine tuned* cancellation in the term $\lambda_1(H_i^u H^{dj}\Sigma_j^i + m' H_i^u H^{di})$.

1.2.3 $SU(5)$ Lagrangian and gauge vertices

The kinetic term for the $SU(5)$ Lagrangian is given by

$$\begin{aligned}\mathcal{L} &= i(\bar{\psi})_a^c(\gamma^\mu D_\mu\psi)^{ca} + i(\bar{\psi})_{ab}(\gamma^\mu D_\mu\psi)^{ab} \qquad (1.44)\\ &= (\bar{\psi})_a^c(i\gamma^\mu\partial_\mu\delta_b^a + \frac{g_5}{\sqrt{2}}\gamma^\mu A_{\mu b}^a)(\psi^c)^b + (\bar{\psi})_{ab}(i\gamma^\mu\partial_\mu\delta_e^b + \frac{g_5}{\sqrt{2}}\gamma^\mu A_{\mu e}^b)(\psi^c)^{ae}\end{aligned}$$

The term involves the X and Y gauge bosons. For a single generation it can be written as

$$\begin{aligned}\mathcal{L}_{X,Y} = &\quad \frac{ig_5}{\sqrt{2}}X_{\mu,i}(\epsilon_{ijk}\bar{u}_k^c\gamma^\mu u_j + \bar{d}_i\gamma^\mu e) \qquad (1.45)\\ &+ \frac{ig_5}{\sqrt{2}}Y_{\mu,i}(\epsilon_{ijk}\bar{u}_k^c\gamma^\mu d_j - \bar{u}_i\gamma^\mu e^c + \bar{d}_i^c\gamma^\mu\nu) + \text{h.c}\end{aligned}$$

The Feynman diagrams which represent the above gauge intarcations are shown in the figures 1.2,1.3 and 1.4.

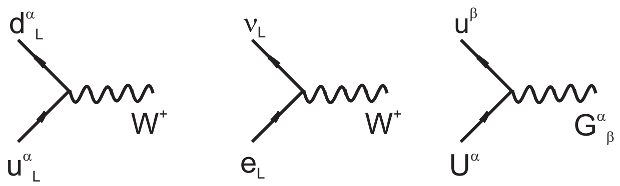


Figure 1.2: Feynman diagrams for the SM gauge boson interactions

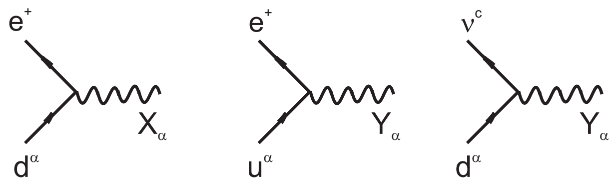


Figure 1.3: Feynman diagrams for vector-like lepto-quark interactions

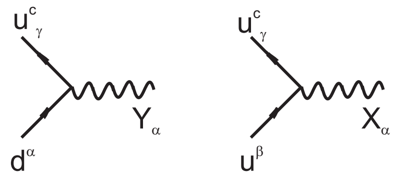


Figure 1.4: Feynman diagrams for diquark interactions

The decays involving leptoquarks (V) violate B-number and L-number. This is:

$$\begin{aligned} V &\rightarrow \bar{l}_L\, u^c_R \qquad B=-1/3 \quad B-L=2/3 \\ V &\rightarrow q_L\, d^c_R \qquad B=2/3 \quad B-L=2/3 \end{aligned} \tag{1.46}$$

Then, $B-L$ is conserved. This feature was in the past a motivation to GUT bariogenesis [Chen 2006], but it was found that generating the sufficient B asymmetry requires high reheating temperature.

1.2.4 Coupling constants and renormalization group

The SM makes predictions for energies below 1 TeV. At these energies interactions are not unified, there are three unrelated coupling constants g_s, g and g' for $SU(3)_C$, $SU(2)_L$ and $U(1)_Y$ respectively. In this theory there is no a explanation for the different strengths of this couplings.

The idea of a Grand Unified Theory is that all interactions are described by one single coupling constant. The three gauge couplings of the SM arise after the $SU(5)$ symmetry breaking. As X and Y acquire masses, the renormalisation group is decoupled into three subgroups.

In 1974, Georgi, Quinn and Weinberg [Georgi et al 1974] observed that the coupling constants behave in a scale-dependent way and vary according to the energy scale in such a way that they must be equal to the $SU(5)$ coupling constant at some energy scale M_U. In 1975, Appelquist and Carazone [Appelquist-Carazone 1975] presented what is known as the *decoupling theorem* which states that if a gauge invariant (under G) Lagrangian field theory contains particles with two very different mass scales ($m \ll M$) and is described by a renormalizable Lagrangian, the behaviour of the light particles in the theory can be described completely by a renormalizable Lagrangian that involves only these particles.

Because of this theorem, gauge couplings are able to be described, at the SM scale, by the β-functions corresponding to $SU(3)_C$, $SU(2)_L$ and $U(1)_Y$ without taking into account $SU(5)$:

$$\frac{dg_i}{dt} = \beta_i(g_i) \equiv b_i \frac{g_i^3}{16\pi^2} \tag{1.47}$$

with $i = 1, 2, 3$ and $t = \ln\mu$, μ is the energy scale.

The relation between the SM and the $SU(5)$ gauge couplings can be obtained by a direct comparison of the covariant derivatives:

$$D_\mu = \partial_\mu + ig_s \sum_{\alpha=1}^{8} G_\mu^\alpha \frac{\lambda^\alpha}{2} + ig \sum_{r=1}^{3} W_\mu^r \frac{\tau^r}{2} + ig' B_\mu \frac{Y}{2} \tag{1.48}$$

$$D_\mu = \partial_\mu + ig_5 \sum_{a=1}^{24} A_\mu^a \frac{L^a}{2} \tag{1.49}$$

The definition of the coupling constants is determined by the normalization of the generators.

In $SU(5)$, $g_5 = g_s = g = g_1$ where g_1 corresponds to the $U(1)$ coupling at the $SU(5)$ scale. Non-Abelian groups have fixed coupling; therefore, the comparison between the two covariant derivatives is made on the Abelian part.

$$ig_1 L^{12} A_\mu^{12} \equiv ig' Y B_\mu \tag{1.50}$$

As A_μ^{12} has been assigned to B_μ, from the definition of hypercharge and L^{12}:

$$\begin{aligned} g_1 L^{12} &= g' Y \\ g_1 \sqrt{\frac{3}{5}} &= g' \end{aligned} \tag{1.51}$$

And this can be used to compute the weak angle (or Weinberg angle) [Georgi et al 1974]:

$$\begin{aligned} \sin^2\theta_W &= \frac{g'^2}{g^2 + g'^2} \\ &= \frac{\frac{3}{5}g^2}{g^2 + \frac{3}{5}g^2} \\ &= \frac{3}{8} \end{aligned} \tag{1.52}$$

Coming back to equation (1.47), for the SM we have (without Higgs boson contributions)

$$\begin{aligned} b_1 &= \frac{2}{3}N_f \\ b_2 &= -\left(\frac{22}{3} - \frac{2}{3}N_f\right) \\ b_3 &= -\left(11 - \frac{2}{3}N_f\right) \end{aligned} \tag{1.53}$$

Where N_f is the number of fermions. If we define $\alpha_i = g_i^2/4\pi$, we obtain

$$\frac{d\alpha_i}{dt} = \frac{1}{2\pi} g_i \frac{dg_i}{dt} \tag{1.54}$$

and therefore

$$\frac{d\alpha_i}{dt} = b_i \frac{\alpha_i^2}{2\pi} \tag{1.55}$$

With these equations and the boundary conditions $\alpha_1(M_X) = \alpha_2(M_X) = \alpha_3(M_X) = \alpha_U$ we can compute the values for α_i at M_W scale. In order to do it, we recall that $\alpha_3(M_W) = \alpha_s(M_W)$, $\alpha_2(M_W) = \alpha_W(M_W)$, $\frac{3}{5}\alpha_1(M_W) = \frac{g'^2(M_W)}{4\pi}$ and the first part of equation (1.52).

Solving the equations, we get

$$\frac{1}{\alpha_i(\mu)} = \frac{1}{\alpha_U} + \frac{b_i}{2\pi}\ln\left(\frac{M_X}{\mu}\right) \tag{1.56}$$

From solutions for $i = 1, 2$ we can obtain

$$\frac{1}{\alpha_2(\mu)} - \frac{1}{\alpha_1(\mu)} = -\frac{1}{2\pi}\frac{22}{3}\ln\left(\frac{M_X}{\mu}\right) \tag{1.57}$$

Inserting this result in the definition of the weak angle, it is straightforward obtain

$$\sin^2\theta_W \approx \frac{3}{8} - \frac{55\alpha(\mu)}{24\pi}\ln\frac{M_X}{\mu} \tag{1.58}$$

We can note that for the limit $\mu = M_X$, we return to result in equation (1.52). This result is for the case in which we have not included the Higgs doublet. When it is include the coefficient $\frac{55}{24\pi}$ is replaced by $\frac{109}{48\pi}$ [Mohapatra 2002].

Now, we shall compute the value of the weak angle at the M_W scale. We take $\alpha^{-1}(M_W) \simeq 128$ and $M_X = 2.1 \times 10^{14} \times 1.5^{\pm 1}\frac{\Lambda}{0.16\text{GeV}}$ where Λ is the QCD scale parameter that denotes the scale at which the QCD interactions are strong. A plausible value for such parameter is given in [Mackenzie et Lepage 1981]: $\Lambda = 160^{+100}_{-80}$ MeV. Then,

$$\sin^2\theta_W = 0.214 \pm 0.003 \pm 0.0028 \tag{1.59}$$

but from PDG data, the experimental value is $\sin^2\theta_W = 0.23152 \pm 0.00014$ [PDG 2006].

Evidently, this result does not coincide with the obtained before, this is because the condition of unification of coupling constants at GUT scale is not true. The interpolation of experimental data has shown that $SU(5)$ without supersymmetry does not unify the interactions at any scale. This can be seen in figure 1.5

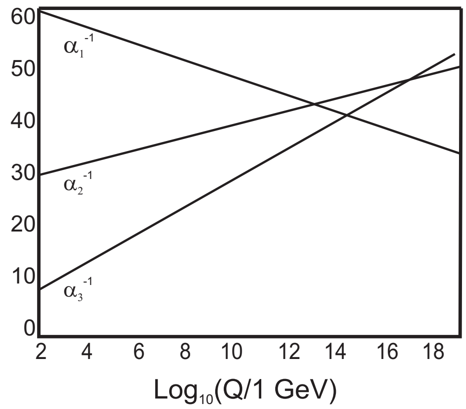


Figure 1.5: Evolution of the α^{-1} functions

1.2.5 Proton decay

One of the most important features of the $SU(5)$ model and any of GUT theory is the prediction of proton decay which arises from a set of tree-level X-boson exchange. The Lagrangian involved in these processes is given by

$$\begin{aligned}\mathcal{L}_{X,Y} &= \frac{1}{\sqrt{2}} g_5 (X_{\mu,\alpha}[\epsilon^{\alpha\beta\gamma}\bar{u}^c_\gamma\gamma^\mu u_\beta + \bar{d}_\alpha\gamma^\mu e^+ - \bar{e}\gamma^\mu d^c_\alpha] \\ &\quad + Y_{\mu,\alpha}[\epsilon^{\alpha\beta\gamma}\bar{u}^c\gamma^\mu d_\beta + \bar{u}_\alpha\gamma^\mu e^+ - \bar{\nu}_e\gamma^\mu d^c_\alpha]\end{aligned} \tag{1.60}$$

From the expression (1.60) it is possible to find a set of $\Delta B = 1$ process. Some of these are shown in figure 1.6. The effective Lagrangian for these processes is

$$\mathcal{L}_{\Delta B=1} = \frac{g_5^2}{2M_X^2}\epsilon^{\alpha\beta\gamma}\epsilon^{ab}(\bar{u}^c_\gamma\gamma^\mu q_{\beta a})(\bar{d}^c_\alpha\gamma_\mu l_b + e^+\gamma_\mu a_{\alpha b}), \tag{1.61}$$

where $X_a = (X, Y)$, $q_a = (u, d)$ and $l_a = (\nu, e)$. It is noted that $\Delta(B - L) = 0$ is a symmetry of the effective Lagrangian. Then, $SU(5)$ contains some tow-body channels for proton decay [Mohapatra 2002]:

$$
\begin{aligned}
BR(p &\rightarrow e^+\pi^0) : 0.40 - 0.60 \\
BR(p &\rightarrow e^+\omega) : 0.05 - 0.20 \\
BR(p &\rightarrow \bar{\nu}\pi^+) : 0.16 - 0.24
\end{aligned}
\tag{1.62}
$$

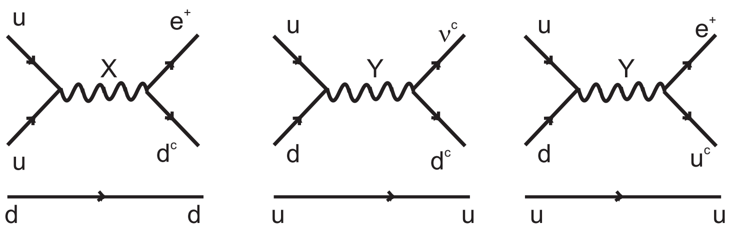


Figure 1.6: Some mechanisms for proton decay in $SU(5)$.

When the proton decay transition amplitude elements are computed from equation (1.61), τ_p is found to be proportional to M_X^2, specifically [Mohapatra 2002]:

$$\tau_p = 2.5 \times 10^{28}\,\text{yr} \qquad \text{to } 1.6 \times 10^{30}\,\text{yr} \tag{1.63}$$

But the current experimental limit is [PDG 2006]

$$\tau_{p \rightarrow e^+\pi^0} \geq 5.1 \times 10^{33}\text{yr}. \tag{1.64}$$

This result rules out $SU(5)$ as a realistic theory.

1.3 Supersymemtric $SU(5)$ model

1.3.1 What is SUSY?

Supersymmetry is a symmetry relating fermions and bosons [Wess and Bagger, 1992], [Martin 1999]. A supersymmetric transformation turns boson states into fermionic ones and viceversa:

$$Q|boson\rangle = |fermion\rangle, \qquad Q|fermion\rangle = |boson\rangle \tag{1.65}$$

From the Haag-Lopuszanski-Sohnius theorem [Coleman et al 1967-1975] this symmetry can be only a direct sum of an internal symmetry algebra and Poincare algebra. The resulting algebra is the Poincare one plus the relations

$$
\begin{aligned}
\{Q, Q^\dagger\} &\sim P^\mu \\
\{Q, Q^\dagger\} &= \{Q, Q^\dagger\} = 0 \\
Q\,P^\mu - P^\mu\,Q &= Q^\dagger\,P^\mu - P^\mu\,Q^\dagger = 0
\end{aligned}
\tag{1.66}
$$

Names	spin 0	spin 1/2	$(SU(3), S(2))_Y$
Q	$(\tilde{u}_L\ \tilde{d}_L)$	$(u_L\ d_L)$	$(3,2)_{1/3}$
$\bar{u}$	$\tilde{u}^*_R$	$u^\dagger_R$	$(3^*,1)_{-4/3}$
$\bar{d}$	$\tilde{d}^*_R$	$d^\dagger_R$	$(3^*,1)_{2/3}$
L	$(\tilde{\nu}\ \tilde{e}_L)$	$(\nu\ e_L)$	$(1,2)_{-1}$
$\bar{e}$	$\tilde{e}^*_R$	$e^\dagger_R$	$(1,1)_2$
H_u	$(H^+_u\ H^0_u)$	$(\tilde{H}^+_u\ \tilde{H}^0_u)$	$(1,2)_1$
H_d	$(H^0_d\ H^+_d)$	$(\tilde{H}^0_d\ \tilde{H}^-_d)$	$(1,2)_{-1}$

Table 1.2: The MSSM Supermultiplets

Names	spin 1/2	spin 1	$(SU(3), S(2))_Y$
gluino, gluon	$\tilde{g}$	g	$(8,1)_0$
winos, W bosons	$\tilde{W}^\pm\ \tilde{W}^0$	$W^\pm\ W^0$	$(1,3)_0$
bino, B boson	$\tilde{B}^0$	B^0	$(1,1)_0$

Table 1.3: The MSSM Gauge supermultiplets

Q and $Q^\dagger$ commute with the SM generators; therefore, the SM eigenstates are also Q and $Q^\dagger$ eigenstates. The irreducible representations of this algebra are called *supermultiplets*. Each supermultiplet contains both fermion and boson states. These are called *superpartners*. The names for the spin-0 partner of quarks and leptons are called *squarks* and *sleptons* respectively and are represented by adding a $\sim$ over the fermion symbol: $\tilde{\nu}_e$, $\tilde{u}_R$, $\tilde{e}_L$, etc. In the Minimal Supersymmetric Standard Model (MSSM) there are two Higgs supermulfields (each with $Y = \pm 1$) in order to avoid triangular gauge anomalies and because of the holomorphy of the superpotential. On the other hand, the gauge bosons have their superpartners called *gauginos*. The content of chiral and gauge supermultiplets of the MSSM is shown in tables 1.2 and 1.3.

One important fact about SUSY is that none of the spartners of the SM particles has been discovered yet and SUSY is a softly broken symmetry.

1.3.2 SUSY $SU(5)$

To construct the supersymmetric extension of the $SU(5)$ model, we need two Higgs superfields H^u and H^d which belong to 5^{ϕ_u} and $\bar{5}^{\phi_d}$ representations. We use a 24-dimensional superfield Σ to break the $SU(5)$ gauge symmetry.

The superpotential for the model is given by [Mohapatra 2002]

$$W = y_u \epsilon^{ijklm} \psi_{ij} \psi_{kl} H^u_m + y_d \psi_{ij} \psi^i H^{dj} + x \operatorname{Tr} \Sigma^2 + \lambda_1 (H^u_i H^{dj} \Sigma^i_j + m' H^u_i H^{di}) \quad (1.67)$$

Since SUSY involves new fields, there will be new contributions to the β function that will affect the evolution of the coupling constants. For supersymmetric

$SU(N)$ the β function is given by [Martin 1999]

$$\beta_i(g_i) = b_{0i}\frac{g_i^3}{16\pi^2} \tag{1.68}$$

where $b_{0i} = -3N + N_f + N_H$. The changes in the contributions is caused by the supersymmetric partners. The evolution for coupling constant within this context is shown in figure 1.7. The new content of particles affects the evolution of gauge coupling as well the GUT scale. The new unification scale is given by

$$M_{X,Susy} \simeq 6 \times 10^{16}\,\text{GeV} \tag{1.69}$$

.

Then, we have that M_X is larger and, therefore, the proton lifetime acquires a new value: $\tau_p \simeq 10^{34}$ yr. Thus, this model is consistent with experimental data.

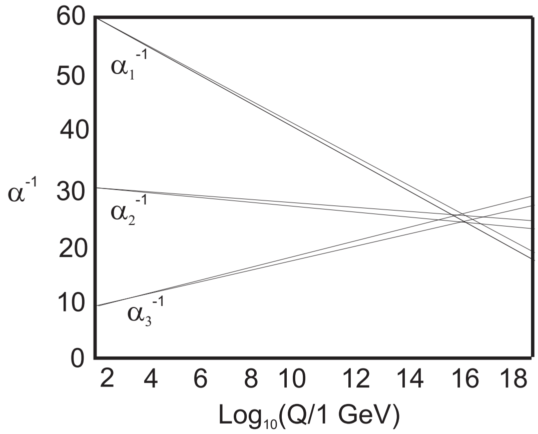


Figure 1.7: Evolution of α^{-1} within the MSSM.

SUSY $SU(5)$ still suffer the so-called the *doublet triplet splitting* problem that we have mentioned above.

Chapter 2

Horizontal symmetries and Froggatt-Nielsen mechanism

"Physics should be beautiful."
Sir F. Hoyle

2.1 The fermion mass hierarchy problem

An important fact from experimental data is that there is a large hierarchy among the particle masses belonging to different generations. That is $m_u \ll m_c \ll m_t$. This is the fermion mass hierarchy problem: there is no explanation for neither this hierarchy nor the quark mixing pattern.

The most general Yukawa Lagrangian in the SM has the following form [Pich 1994]:

$$\mathcal{L}_Y = y^d_{jk}\bar{Q}'_{Lj}\Phi d'_{Rk} + y^u_{jk}\bar{Q}'_{Lj}\bar{\Phi}u'_{Rk} + y^l_{jk}\bar{L}'\Phi l'_{Rk} \tag{2.1}$$

where we are using the notation of equation (1.1) and Φ represents the Higgs field. The indices run over the three generations and the prime indicates that these fields are written in the gauge interaction basis.

After the SSB this Lagrangian becomes

$$\mathcal{L}_Y = \bar{\mathbf{d}}'_L\mathcal{M}'_d\mathbf{d}'_R + \bar{\mathbf{u}}'_L\mathcal{M}'_u\mathbf{u}'_R + \bar{\mathbf{l}}'_L\mathbf{M}'_l\mathbf{l}'_R \tag{2.2}$$

Here, the corresponding mass matrices are given by

$$(\mathcal{M}'_d)_{ij} \equiv -y^d_{ij}v/\sqrt{2}, \qquad (\mathcal{M}'_u)_{ij} \equiv -y^u_{ij}v/\sqrt{2}, \qquad (\mathcal{M}'_d)_{ij} \equiv -y^l_{ij}v/\sqrt{2} \tag{2.3}$$

The diagonalization of these matrices yield the quark and charged lepton masses.

$\mathcal{M}'_u$ is not an Hermitian matrix but $\mathcal{M}'_u\mathcal{M}'^\dagger_u$ do is. Then the latter can be diagonalized by an unitary matrix. The same occurs with $\mathcal{M}'^\dagger_u\mathcal{M}'_u$. Let $\mathcal{U}_L$ and $\mathcal{U}_R$ be two matrices such that they diagonalize the last two Hermitian matrices respectively. Thus, the matrix $\mathcal{M}'_u$ is diagonalized as

$$\mathcal{U}_L\mathcal{M}'_u\mathcal{U}^\dagger_R = \mathcal{M}_u \tag{2.4}$$

In the same way, there are two matrices $\mathcal{D}_L$ and $\mathcal{D}_R$ that diagonalize $\mathcal{M'}_d$:

$$\mathcal{D}_L \mathcal{M'}_d \mathcal{D}_R^\dagger = \mathcal{M}_d \tag{2.5}$$

Now, inserting these matrices in the Lagrangian we obtain the Yukawa Lagrangian in the with mass eigenstate basis.

$$\begin{aligned}\mathcal{L}_Y &= \bar{\mathbf{d}}'_L \mathcal{D}_L^\dagger \mathcal{D}_L \mathcal{M'}_d \mathcal{D}_R^\dagger \mathcal{D}_R \mathbf{d}'_R + \bar{\mathbf{u}}'_L \mathcal{U}_L^\dagger \mathcal{U}_L \mathcal{M'}_u \mathcal{U}_R^\dagger \mathcal{U}_R \mathbf{u}'_R + \text{h.c.}\\ &= \bar{\mathbf{d}}_L \mathcal{M}_d \mathbf{d}_R + \bar{\mathbf{u}}_L \mathcal{M}_u \mathbf{u}_R + \text{h.c.}\\ &= \bar{\mathbf{d}} \mathcal{M}_d \mathbf{d} + \bar{\mathbf{u}} \mathcal{M}_u \mathbf{u}\end{aligned} \tag{2.6}$$

where the mass eigenstates are defined

$$\mathbf{d}_{L,R} \equiv \mathcal{D}_{L,R} \mathbf{d}'_{L,R} \qquad \mathbf{u}_{L,R} \equiv \mathcal{U}_{L,R} \mathbf{u}'_{L,R} \tag{2.7}$$

The charged current Lagrangian of SM has the form

$$\mathcal{L} = \frac{g}{2\sqrt{2}} W_\mu^\dagger [\bar{\mathbf{u}}' \gamma^\mu (1-\gamma_5) \mathbf{d}' + \bar{\nu}' \gamma^\mu (1-\gamma_5) \mathbf{l}'] + \text{h.c.} \tag{2.8}$$

Inserting the mass eigenstates, this Lagrangian becomes

$$\mathcal{L} = \frac{g}{2\sqrt{2}} W_\mu^\dagger [\bar{\mathbf{u}} \gamma^\mu (1-\gamma_5) \mathbf{V} \mathbf{d} + \bar{\nu} \gamma^\mu (1-\gamma_5) \mathbf{l}] + h.c. \tag{2.9}$$

Here, $\mathbf{V}$ is a mixing matrix called the Cabibbo-Kobayashi-Maskawa (CKM) matrix and it is given by

$$\mathbf{V} \equiv \mathcal{U}_L \mathcal{D}_L^\dagger \tag{2.10}$$

We note that this matrix mixes any up-type quark with all down-type quarks. Also, it can be noted that there is no mixing among leptons, this is because in the SM neutrinos are massless; so, it is always possible to rotate the neutrino fields in such a way that the charged lepton mass matrix remains diagonal.

The CKM matrix can be parametrised with three mixing angles and one phase, [PDG 2006] $R(\theta_{12})R(\theta_{13},\delta_{13})R(\theta_{23})$:

$$\mathbf{V} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{13}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{-i\delta_{13}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{-i\delta_{13}} & s_{23}c_{13} \\ s_{12}c_{23} - c_{12}s_{23}s_{13}e^{-i\delta_{13}} & -c_{12}c_{23} - s_{12}s_{23}s_{13}e^{-i\delta_{13}} & c_{23}c_{13} \end{pmatrix} \tag{2.11}$$

where $c_{ij} \equiv \cos(\theta_{ij})$ and i, j are generation indexes. Experiments have shown that $s_{13} \ll s_{23} \ll s_{12}$.

Another parametrization, given by Wolfenstein, exhibit explicitly the mass hierarchy:

$$s_{12} = \lambda \qquad s_{23} = A\lambda^2 \qquad s_{13}e^{i\delta} = A\lambda^3(\rho + i\eta) \tag{2.12}$$

In this way, the CKM matrix takes the form

$$\mathbf{V} = \begin{pmatrix} 1-\lambda^2/2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\lambda^2/2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & A-\lambda^2 & 1 \end{pmatrix} \tag{2.13}$$

Since this matrix is unitary, $\mathbf{V}_{ij}\mathbf{V}^*_{jk} = \delta_{ik}$, a common unitary triangle arises from $\mathbf{V}_{ud}\mathbf{V}^*_{ub} + \mathbf{V}_{cd}\mathbf{V}^*_{cb} + \mathbf{V}_{td}\mathbf{V}^*_{tb} = 0$. This relation, in terms of $(\bar{\rho}), \bar{\eta}$ is used to present measurements of CKM elements [PDG 2006].

The SM quark masses at M_Z scale take the following values [Aristizabal 2003], [PDG 2006]:

$$\begin{aligned} m_u &= 2.33^{+0.42}_{-0.45}\ \text{MeV}, \quad m_c = 677^{+56}_{-61}\ \text{MeV}, \quad m_t = 181 \pm 13\ \text{GeV} \\ m_d &= 4.69^{+0.60}_{-0.66}\ \text{MeV}, \quad m_s = 93.4^{+11.8}_{-13.0}\text{MeV}, \quad m_b = 3.00 \pm 11\ \text{GeV} \end{aligned} \tag{2.14}$$

The relations among the masses of the different quark generations (at M_Z scale) can be expressed in powers of $\lambda = 0.22$:

$$\begin{aligned} \frac{m_u}{m_c} &= 0.0034 \sim \lambda^3 & \frac{m_d}{m_s} &= 0.0521 \sim \lambda^2 \\ \frac{m_b}{m_t} &= 0.0165 \sim \lambda^2 & \frac{m_c}{m_t} &= 0.0037 \sim \lambda^3 \\ \frac{m_s}{m_b} &= 0.0311 \sim \lambda^2 \end{aligned} \tag{2.15}$$

2.2 Horizontal symmetries

The fermion mass hierarchy can be explained through the *Froggatt-Nielsen Mechanism* [Froggatt and Nielsen 1979]. According to this approach, a global$H$ symmetry spontaneously broken by the VEV of an scalar singlet is responsible of the observed fermion mass pattern.

Let H be a symmetry under which fields transforms as [Leurer et al 1992]:

$$\begin{aligned} Q &\to LQ & \bar{d}_R &\to R_d\bar{d}_R \\ \bar{u}_R &\to R_u\bar{u}_R & \Phi &\to P\Phi \end{aligned} \tag{2.16}$$

In the mass basis:

$$\bar{u}_L \to L_u\bar{u}_L \qquad \bar{d}_L \to L_d\bar{d}_L \tag{2.17}$$

where $L_u = \mathcal{U}_L L \mathcal{U}_L^\dagger$ and $L_d = \mathcal{D}_L L \mathcal{D}_L^\dagger$ such that $L_d = \mathbf{V}^\dagger L_v \mathbf{V}$.

The symmetry H is found to be broken: In fact, if H were unbroken, from equation (2.1) and from the form in which fields transform, we would have:

$$L\mathcal{M}'_d R_d = \mathcal{M}'_d, \qquad L\mathcal{M}'_u R_u = \mathcal{M}'_u. \tag{2.18}$$

Then, it is straightforward to show that

$$L\mathcal{M'}_d\mathcal{M'}^{\dagger}_d L^{\dagger} = \mathcal{M'}_d\mathcal{M'}^{\dagger}_d, \qquad L\mathcal{M'}_u\mathcal{M'}^{\dagger}_u L^{\dagger} = \mathcal{M'}_u\mathcal{M'}^{\dagger}_u. \tag{2.19}$$

Or in the mass basis

$$[L_d, \mathcal{M}^2_d] = [L_u, \mathcal{M}^2_u] = 0. \tag{2.20}$$

This result says that if the symmetry were no broken there were degeneracy in the quark sector. L_u and L_d were forced to be diagonal, but because they are related by $L_d = \mathbf{V}^{\dagger}L_v\mathbf{V}$ they would have the same eigenvalues. If the eigenvalues were different, i.e. there were no degeneracy, then all mixing angles were indeed null. So we can note that as there is no degeneracy in the quark sector and as the three quark generations mix, H must be a broken symmetry.

As an explicit example, we take the SM and we consider an Abelian continuous symmetry $H = U(1)_F$ at a very high energy scale. Once $U(1)_F$ is spontaneously broken by the VEV of a SM singlet scalar S, a set of effective operators are induced by the integration of heavy fields of mass M_F (much larger that electroweak SSB scale). These heavy fields couple to the SM fermions and Higgs boson as well. The breaking of $U(1)_F$ occurs at a scale $\Lambda_F \simeq \langle S\rangle$. The hierarchy among fermions is provided by the hierarchy among the effective operators which are suppressed by powers of the parameter $\lambda = \langle S\rangle/M_F$ [Duque 2006].

The Yukawa terms in the SM Lagrangian have the form

$$\mathcal{L}_{\text{Yukawa}} = y^d_{ij}Q_i\phi_d\bar{d}_j + y^u_{ij}Q_i\phi_u\bar{u}_j + h.c. \tag{2.21}$$

At the scale in which H is an unbroken symmetry, for example, to produce up quarks terms of order 2, the Yukawa Lagrangian contains the *Froggatt-Nielsen heavy fields* R and T. To write it, we use the *universality principle* that supposes $y^d \approx 1$ and $y^u \approx 1$. Then, we have

$$\mathcal{L}_Y^{\text{H exact}} = \bar{Q}_{F(\bar{Q})}\tilde{\phi}_0 R_{F(R)} + \bar{R}_{F(R)}S_{-1}T_{F(T)} + \bar{T}_{F(\bar{T})}S_{-1}u_{F(u)} \tag{2.22}$$

where the subindexes indicate the horizontal charge assignment. We assign the charges $F(\phi) = F(\tilde{\phi}) = 0$ and $F(S) = -1$ and from this we can constrain the charges of the other fields in such way that all the terms remain invariant under H.

$$\begin{aligned} F(\bar{Q}) + F(R) &= 0 \\ F(\bar{R}) + F(T) &= 1 \\ F(\bar{T}) + F(u) &= 1 \end{aligned} \tag{2.23}$$

A diagram for the horizontal interactions with two heavy fields is shown in figure 2.1.

According to the Feynman rules [Peskin 1995], the expression for the propagators in the diagram 2.1 has the form

$$Q\phi\frac{i(\gamma_\mu p^\mu_R + M_F)}{p^2_R + M^2_F}S\frac{i(\gamma_\mu p^\mu_T + M_F)}{p^2_T + M^2_F} \tag{2.24}$$

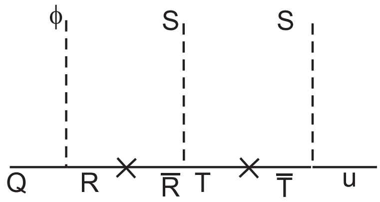


Figure 2.1: SM horizontal symmetry diagram

i	F($\bar{Q}$)	F(u)	F(d)	F($\bar{L}$)	F(e)
1	3	5	3	1	6
2	2	2	2	1	6
3	0	0	2	1	1

Table 2.1: Charges for three fermion generations

The above expression produces the effective term once H is broken through $\langle S \rangle$,

$$\mathcal{L}_{\rm eff} = (\frac{\langle S \rangle}{M_F})^2 \bar{Q}\tilde{\phi}u + \text{h.c.}. \tag{2.25}$$

A general effective Lagrangian that includes the three fermion generations has the form

$$\mathcal{L}_{\rm eff} = \lambda^{n_{ij}} \bar{Q}_i\tilde{\phi}u_j + \lambda^{m_{ij}} \bar{Q}_i\tilde{\phi}d_j + \lambda^{s_{ij}} \bar{L}_i\tilde{\phi}e_j + \text{h.c.} \tag{2.26}$$

where $n_{ij} = F(\bar{Q}_i) + F(u_j)$,$m_{ij} = F(\bar{Q}_i) + F(d_j)$ and $s_{ij} = F(\bar{L}_i) + F(e_j)$ and the mass matrices

$$\begin{aligned} \mathcal{M}^u_{ij} &= v\lambda^{F(\bar{Q}_i)+F(u_j)} \\ \mathcal{M}^d_{ij} &= v\lambda^{F(\bar{Q}_i)+F(d_j)} \\ \mathcal{M}^l_{ij} &= v\lambda^{F(\bar{L}_i)+F(e_j)} \end{aligned} \tag{2.27}$$

Now, we wonder if it is possible to find a set of charges that correctly fit the measured masses and mixing angles.

We have found that with the charge assignment shown in the table 2.1 this can be achieved.

It is straightforward to find the mass matrices

$$\begin{aligned}
\mathcal{M}^u &= v \begin{pmatrix} \lambda^8 & \lambda^5 & \lambda^3 \\ \lambda^7 & \lambda^4 & \lambda^2 \\ \lambda^5 & \lambda^2 & 1 \end{pmatrix} \\
\mathcal{M}^d &= v\lambda^2 \begin{pmatrix} \lambda^4 & \lambda^3 & \lambda^3 \\ \lambda^3 & \lambda^2 & \lambda^2 \\ \lambda & 1 & 1 \end{pmatrix} \\
\mathcal{M}^l &= v\lambda^2 \begin{pmatrix} \lambda^5 & \lambda^2 & 1 \\ \lambda^5 & \lambda^2 & 1 \\ \lambda^5 & \lambda^2 & 1 \end{pmatrix}
\end{aligned} \tag{2.28}$$

If we take $\lambda \approx 0.22$, we can note that the quark matrices fit approximately to the expected values for relations among the generations. Then, we can conclude that the Froggatt-Nielsen mechanism can provide approximately the hierarchy for SM fermion masses. However, this method can only show the magnitude order of the entries but we can not compute the coefficients of each one. In the next chapter we shall modify this method in order to properly compute these coefficients in a supersymmetric $SU(5)$ extended with a horizontal symmetry.

Chapter 3
$SU(5) \times U(1)_H$ Model

"Science is facts; just as houses are made of stones,
so is science made of facts; but a pile of stones is not a house
and a collection of facts is not necessarily science."
Henri Poincare

In the last chapter we saw that the Froggatt-Nielsen mechanism can explain the hierarchy fermion mass of the SM. In this chapter we will consider the SUSY $SU(5) \times U(1)_H$ model. This model has been previously studied in references [Aristizabal 2003], [Duque 2006] and [Nardi et al 2007].

In this model, a horizontal Abelian flavor symmetry unbroken above the energy scale at which $SU(5)$ is still an exact symmetry is implemented. Recently, Chen et al. have studied some features of such a symmetry in $SU(5)$ [Chen 2008]. The Abelian flavor symmetry, in that case, was broken through the VEV of a $SU(5)$ scalar singlet as the one discussed in the last chapter. As a result, they arrived at the mass hierarchy structure but without specifying coefficients of the entries in the mixing matrix. Our goal is to get these coefficients by using a H-charged $SU(5)$ adjoint representation to break the Abelian symmetry. This idea was proposed by Enrico Nardi and Diego Aristizábal from Universidad de Antioquia [Nardi et al 2007], [Aristizabal and Nardi 2004]. The main motivation for this is found in the Georgi-Jarlskog's article [Georgi and Jarlskog 1979]. They used a 45 Higgs representation field in order to explain the difference in the lepton-quark mass ratios for the three fermion generations.

3.1 The mass hierarchy at GUT scale

In 2007, Ross and Serna have given the mass structure for fermions at GUT scale [Ross-Serna 2007]. They took as starting point the values of fermion masses at a low energy scale and used the RG equations for different choices of tanβ (tan$\beta = \langle H_u^0 \rangle \,/\, \langle H_d^o \rangle$). The ratios in which we are concerned are

$$\begin{aligned} \frac{m_u}{m_c} &\approx 0.0027(6) \qquad \frac{m_c}{m_t} \approx 0.0025(2) \\ \frac{m_d}{m_s} &\approx 0.051(7) \qquad \frac{m_s}{m_b} \approx 0.014(4) \\ \frac{m_e}{m_\mu} &\approx 0.0048(2) \qquad \frac{m_\mu}{m_\tau} = 0.059(2). \end{aligned} \tag{3.1}$$

This is the GUT version of equation (2.15).

We can express this structure using a parameter $\epsilon = \frac{1}{25} \approx \lambda^2$.

$$m_u : m_c : m_t = \epsilon^4 : \epsilon^2 : 1 \qquad m_d : m_s : m_b = \epsilon^3 : \epsilon^2 : \epsilon \tag{3.2}$$

The numerical V_{CKM} at the GUT scale is given by [Fusaoka-Koide 1997]

$$V(M_X) = \begin{pmatrix} 0.9754 & 0.2205 & -0.0026 \\ -0.2203 & 0.9749 & 0.0318 \\ 0.0075 & 0.0311 & 0.9995 \end{pmatrix} \tag{3.3}$$

3.2 H-charge assignments Constraints

3.2.1 Anomaly cancellation

We must ensure that our model is anomaly free in what regards the horizontal charge. This condition is expressed as

$$\begin{aligned} \mathcal{A} &= \sum_{i=1}^{3} (H(\bar{5}_i) + 3H(10_i)) + H(\bar{5}^{\phi_d}) + H(5^{\phi_u}) = 0 \\ &= \sum_{i=1}^{3} (f_i + 3t_i) + f_d + f_u = 0 \end{aligned} \tag{3.4}$$

where $H(\bar{5}_i) = f_i$, $H(10_i) = t_i$ and f_d, f_u are the H-charge for the Higgs fields [Nardi et al 2007].

This condition can be redefined by a set of shifts in the charges

$$\begin{aligned} f_i &\to f_i + a \\ t_i &\to t_i + b \\ f_d &\to f_d - (a+b) \\ f_u &\to f_u - 2b \end{aligned} \tag{3.5}$$

such that the new anomaly term is

$$\mathcal{A}' = \mathcal{A} + 2(a+3b) \tag{3.6}$$

The model requires the invariance of the term $M_\phi \bar{\phi}^d \phi^u$ under H. Then, $a+3b = 0$ and we have the following constraint:

$$b = -\frac{1}{3} a \tag{3.7}$$

	5_1	5_2	5_3	10_1	10_2	10_3
(1)	1	1	1	2	1	0
(2)	-1	-3	-1	-2	1	0
(3)	-5	1	1	2	1	0
(4)	5	-3	-1	-2	1	0
(5)	1	-3	1	2	1	0
(6)	-1	1	-1	-2	1	0
(7)	-5	-3	1	2	1	0
(8)	5	1	-1	-2	1	0
(9)	1	3	-1	2	-1	0
(10)	-1	-1	1	-2	-1	0
(11)	-5	3	-1	2	-1	0
(12)	5	-1	1	-2	-1	0
(13)	1	-1	-1	2	-1	0
(14)	-1	3	1	-2	-1	0
(15)	-5	-1	-1	2	-1	0
(16)	5	3	1	-2	-1	0

Table 3.1: Possible H charge assignments

3.2.2 Charge assignments

From equation (3.2) and according to the idea of the FN mechanism, the horizontal charges for quarks must follow the equations

$$\begin{aligned} |10_1 + 10_1 + 5^{\phi_u}| &= 4 \qquad |\bar{5}_1 + 10_1 + \bar{5}^{\phi_d}| = 3 \\ |10_2 + 10_2 + 5^{\phi_u}| &= 2 \qquad |\bar{5}_2 + 10_2 + \bar{5}^{\phi_d}| = 2 \\ |10_3 + 10_3 + 5^{\phi_u}| &= 0 \qquad |\bar{5}_3 + 10_3 + \bar{5}^{\phi_d}| = 1 \end{aligned} \tag{3.8}$$

In addition, the term $m'H^dH^u$ in the superpotential (1.67) must be invariant. This leads to the condition

$$f_d + f_u = 0 \tag{3.9}$$

Then, from equations (3.8), (3.4) and (3.9) we have a set of linear equations.

We obtained the set of solutions by choosing the charges to obey the equation (3.8) and $f_d = f_u = 0$. For example, from $|\bar{5}_3 + 10_3 + \bar{5}^{\phi_d}| = 1$ and $|10_3 + 10_3 + 5^{\phi_u}| = 0$ it follows that t_3 must be 0, and for f_3 we have only two possibilities: $f_3 = \pm 1$. After an iterative procedure for each pair of equations we find all possible combinations. The results are shown in table 3.1.

We can discard eight solutions that have the same charges for two equal representations since they generate two rows proportional between them. Later, we discard six solutions that generates entries of order 0 in the down-type quark matrix. Then we have two solutions, one of them does not satisfy the anomaly cancellation. So we have an unique solution which corresponds to solution (11).

f_1	f_2	f_3	t_1	t_2	t_3	f_u
-4	4	0	5/3	-4/3	-1/3	2/3

Table 3.2: Best H charge assignments for fermions in $SU(5) \times U(1)$

We can make a shift such that $b = -1/3$ and in this case the new set of charges is given in table 3.2. We make such a shift in order to avoid invariant terms of the type $\lambda' \psi^{ij} \psi_i \psi_j$ which can spoil proton stability. With this charge selection, the suppression factors in the effective Lagrangian can be written as:

$$y^d_{ij} = Y^d_{ij} \epsilon^{f_i + t_j + f_d}, \qquad y^u_{ij} = Y^u_{ij} \epsilon^{t_i + t_j + f_d}. \tag{3.10}$$

The charge assignment in table 3.2 generates the matrices for up and down type quarks given in equation (3.11), where we assume $Y^{d,u}_{ij}$ to be of order unity.

$$\mathcal{M}'^u \sim \begin{pmatrix} \epsilon^4 & \epsilon^1 & \epsilon^2 \\ \epsilon^1 & \epsilon^2 & \epsilon^1 \\ \epsilon^2 & \epsilon^1 & 1 \end{pmatrix} \qquad \mathcal{M}'^d \sim \begin{pmatrix} \epsilon^3 & \epsilon^6 & \epsilon^5 \\ \epsilon^5 & \epsilon^2 & \epsilon^3 \\ \epsilon^1 & \epsilon^2 & \epsilon^1 \end{pmatrix} \tag{3.11}$$

3.2.3 Forbidden representations

In the matrices shown in equation (3.11) we note that the entries $\mathcal{M}^u_{12}$ and $\mathcal{M}^u_{21}$ are not very suppressed and, therefore, they affect the determinant of the matrix. This problem can be corrected by forbidding the representations involved in the effective Lagrangian of order one.

The two possible diagrams at order one are shown in figure 3.1

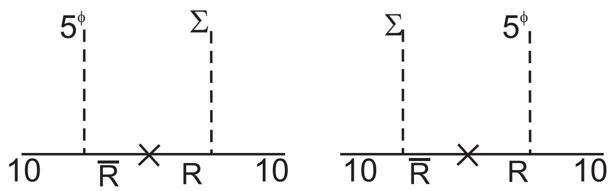


Figure 3.1: Diagrams at order one

For the first diagram, the Lagrangian is

$$\mathcal{L} = 10_{-4/3} 5^{\phi_u}_{2/3} \bar{R}_{2/3} + R_{-2/3} \Sigma_{-1} 10_{5/3}. \tag{3.12}$$

Where the subindexes indicate the horizontal charge assignment.

We will find the representations valid for R.

We know that $10 \otimes 5 = \bar{10} \oplus 40$ and $10 \otimes 24 = 10 \oplus 15 \oplus \bar{40}$. Then, we can assign to R a representation $\bar{10}$ or 40. Thus, the representations $\bar{10}_{-2/3}$ and $40_{-2/3}$ are forbidden.

On the other hand, the Lagrangian for the second diagram is

$$\mathcal{L} = 10_{-4/3}\Sigma_{-1}\bar{R}_{7/3} + R_{-7/3}5^{\phi_u}_{2/3}10_{5/3}. \tag{3.13}$$

And, in this case, the representations $10_{-7/3}$ and $\bar{40}_{-7/3}$ must be forbidden.

In conclusion, we must ignore the following representations:

$$\begin{array}{ll} \bar{10}_{-2/3}, & 40_{-2/3} \\ 10_{-7/3}, & \bar{40}_{-7/3}. \end{array} \tag{3.14}$$

And the new mass matrix $\mathcal{M}'^u$ has the following structure

$$\mathcal{M}'^u \sim \begin{pmatrix} \epsilon^4 & \epsilon^2 & \epsilon^2 \\ \epsilon^2 & \epsilon^2 & \epsilon^1 \\ \epsilon^2 & \epsilon^1 & 1 \end{pmatrix} \tag{3.15}$$

3.2.4 Testing the Up matrix

We want to check if the mass matrix in (3.15) fits the experimental results already discussed in section 3.1. We take this matrix and generate random coefficients in the range $[0.2, 5]$ and obtain the matrix that diagonalize it.

After including the VEV for the Higgs fields, the diagonal matrix becomes

$$\mathcal{M}^u = \begin{pmatrix} 0.004556 & 0 & 0 \\ 0 & 0.504603 & 0 \\ 0 & 0 & 171.737 \end{pmatrix} \tag{3.16}$$

It can be seen m_t fits quite well the experimental data. The matrix used to obtain (3.16) is

$$\mathcal{U}_L = \begin{pmatrix} 0.991066 & -0.133330 & 0.003221 \\ -0.133359 & -0.990412 & 0.036020 \\ 0.001611 & 0.036128 & 0.999345 \end{pmatrix} \tag{3.17}$$

If we take $\mathcal{D}_L \approx 1$, we can see that (3.17) coincides approximately with (3.3). We can also check other constrains as

$$\frac{m_u}{m_c} = -0.00885 \qquad \frac{m_c}{m_t} = 0.00386 \tag{3.18}$$

$$\theta_{12} = -0.13373 \qquad \theta_{23} = 0.03602 \tag{3.19}$$

We can note that the mixing angle θ_{12} has a little value but this is not in contradiction with the experimental data. All these results serve as motivation for computing the explicit coefficients to obtain a realistic CKM matrix.

Product	Irreducible representations
$5 \otimes \overline{5}$	$1 \oplus 24$
$10 \otimes 5$	$\overline{10} \oplus 40$
$10 \otimes \overline{10}$	$1 \oplus 24 \oplus 75$
$10 \otimes 15$	$\overline{45} \oplus \overline{105}$
$15 \otimes 5$	$35 \oplus 40$
$10 \otimes \overline{5}$	$5 \oplus 45$
$15 \otimes \overline{5}$	$5 \oplus 70$
$15 \otimes \overline{15}$	$1 \oplus 24 \oplus 200$
$\overline{40} \otimes \overline{5}$	$45 \oplus \overline{50} \oplus \overline{105}$
$40 \otimes \overline{5}$	$10 \oplus 15 \oplus 175$
$\overline{45} \otimes \overline{5}$	$\overline{10} \oplus 40 \oplus \overline{175}$
$\overline{70} \otimes \overline{5}$	$\overline{15} \oplus \overline{160} \oplus 175$
$24 \otimes \overline{5}$	$\overline{5} \oplus \overline{45} \oplus \overline{70}$
$24 \otimes 10$	$10 \oplus 15 \oplus \overline{40} \oplus 175$
$24 \otimes 15$	$10 \oplus 15 \oplus 160 \oplus 175$
$24 \otimes 40$	$\overline{10} \oplus 35 \oplus 40 \oplus 40 \oplus \overline{175} \oplus \overline{210} \oplus 450$
$24 \otimes 45$	$45 \oplus 45 \oplus 5 \oplus \overline{50} \oplus 70 \oplus \overline{105} \oplus 280 \oplus \overline{480}$
$24 \otimes 70$	$70 \oplus 70 \oplus 5 \oplus 45 \oplus 70 \oplus \overline{105} \oplus 280 \oplus 280 \oplus \overline{450} \oplus \overline{480}$

Table 3.3: Reduction of products of representations

3.3 Product of representations

Before going to the explicit calculation of some propagators, we present in table 3.3 a kind of dictionary of the main reductions of the product of representations of $SU(5)$. Details about how to get these reductions can be found in the appendix.

3.4 The effective operators

Once we have the mass structure, the next step is to obtain the explicit coefficients through a symmetry breaking using a H-charged adjoint representation. The heavy fields are integrated out and the suppression term is given by powers of $\epsilon = \langle V \rangle / M_F$, where $\langle V \rangle$ is the VEV of the adjoint representation.

3.4.1 Computing the effective operators

Quantum field theory gives us the basic tools to compute the diagrams for the horizontal interactions. We use the functional quantization for fermion fields [Pokorski 2000].

The generating functional of the Green's function is given by the path integral

$$W[\eta, \bar{\eta}] = N \int \mathcal{D}\Psi\, \mathcal{D}\bar{\Psi} \exp[i(S[\Psi, \bar{\Psi}] + \int d^4x\, \bar{\eta}(x)\Psi(x) + \bar{\Psi}(x)\eta(x))]. \quad (3.20)$$

Where $N^{-1} = W[0,0]$ and η , $\bar{\eta}$ are Grassmann fields. And the Feynman propagator is given by

$$S_F(x-y) = \langle 0|T\Psi(x)\bar{\Psi}(y)|0\rangle = (i)^{-2}\frac{\delta}{\delta\bar{\eta}(x)}W[\eta, \bar{\eta}]\frac{\overleftarrow{\delta}}{\delta\eta(x)}|_{\eta=\bar{\eta}=0}. \quad (3.21)$$

Propagator for the dimension 5 representation

The simplest propagator for the heavy fields, R and $\bar{R}$, is the corresponding to the 5 dimensional representation. For this one, we have the diagram 3.2. The generating functional is given by

$$W[\bar{\eta}, \eta] = N^{-1} \int \mathcal{D}R\, \mathcal{D}\bar{R} \exp\left[-\mathrm{i}\int d^4x\mathcal{L}_0 + \int d^4x(\bar{R}_a\eta^a + \bar{\eta}_c R^c)\right] \quad (3.22)$$

Now, we have that the propagator $[5]^b_d$ has the expression

$$\begin{aligned}
[5^b_d] &= (i)^{-2}\frac{\delta}{\delta\bar{\eta}_b}W[\eta, \bar{\eta}]\frac{\overleftarrow{\delta}}{\delta\eta^d} \\
&= i\frac{\delta}{\delta\bar{\eta}_b}\int \mathcal{D}R\,\mathcal{D}\bar{R}\exp\left[-\mathrm{i}\int d^4x\mathcal{L}_0 + \int d^4x(\bar{R}_a\eta^a + \bar{\eta}_c R^c)\right](-\bar{R}_m\delta^m_d) \\
&= -\int \mathcal{D}R\,\mathcal{D}\bar{R}\exp\left[-i\int d^4x\mathcal{L}_0 + \int d^4x(\bar{R}_a\eta^a + \bar{\eta}_c R^c)\right](\bar{R}_a R^c\delta^a_d\delta^b_c)_{\eta=\bar{\eta}=0} \quad (3.23)
\end{aligned}$$

We uses the theorem for functional determinants [Pokorski 2000]

$$\int \mathcal{D}R\,\mathcal{D}\bar{R}\exp\left[-i\int d^4x\mathcal{L}_0 + \int d^4x(\bar{R}_a\eta^a + \bar{\eta}_c R^c)\right](\bar{R}_d R^b) = \left(\frac{1}{\not{p} - M + i\epsilon}\right)\delta^b_d \quad (3.24)$$

Then, the propagator takes the form

$$[5^b_d] = -\left(\frac{1}{\not{p} - M + i\epsilon}\right)\delta^b_d \quad (3.25)$$

Then, the effective propagator, when we have $p \ll M$, for a 5 representation is given by

$$5^b\bar{5}_d \to \frac{1}{M}\delta^b_d \quad (3.26)$$

Now, we can compute the effective operator corresponding to the diagram 3.2 and with Lagrangian

$$\mathcal{L} = \bar{5}_l\Sigma^l_b R^b + \bar{R}_d\bar{5}^{\phi_d}_e 10^{de} \quad (3.27)$$

The effective Lagrangian that we obtain is

$$\begin{aligned}
\mathcal{L}_{eff} &= \bar{5}_l\Sigma^l_b([5]^b_d)\bar{5}^{\phi_d}_e 10^{de} \\
&= \frac{1}{M}\bar{5}_l\Sigma^l_d\bar{5}^{\phi_d}_e 10^{de} \quad (3.28)
\end{aligned}$$

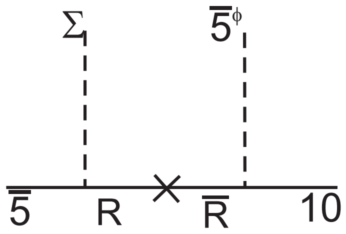


Figure 3.2: Horizontal symmetry for a 5 dimensional representation diagram

Propagators for the dimension $50^{(r)}$ and 45 representations

Now, we shall obtain the effective propagator for the fields in a reducible representation 50_c^{ab}. For this case, the generating functional is

$$W[\bar{\eta},\eta] = \frac{1}{2}\int \mathcal{D}R\,\mathcal{D}\bar{R}\,\exp\left[-\,\mathrm{i}\int d^4x\mathcal{L}_0 + \int d^4x(\bar{R}^p_{qr}\eta^{qr}_p + \bar{\eta}^i_{jk}R^{jk}_i)\right]. \quad (3.29)$$

We take the equation (3.21) to write the propagator as:

$$[R^{ab}_c\bar{R}^n_{lm}] = \frac{\delta}{\delta\bar{\eta}^c_{ab}}W[\eta,\bar{\eta}]\frac{\overleftarrow{\delta}}{\delta\eta^{lm}_n} \quad (3.30)$$

And then

$$\begin{aligned}[R^{ab}_c\bar{R}^n_{lm}] &= (i)^{-2}\frac{\delta}{\delta\bar{\eta}^c_{ab}}\frac{1}{2}\int \mathcal{D}R\,\mathcal{D}\bar{R}\,\exp\left[-\,\mathrm{i}\int d^4x\mathcal{L}_0 + \int d^4x(\bar{R}^p_{qr}\eta^{qr}_p + \bar{\eta}^i_{jk}R^{jk}_i)\right] \\ &\quad\times(-\,\mathrm{i}\,\bar{R}^p_{qr}\delta^n_p(\delta^q_l\delta^r_m - \delta^q_m\delta^r_l)) \\ &= \frac{1}{2}\int \mathcal{D}R\,\mathcal{D}\bar{R}\,\exp\left[-\,\mathrm{i}\int d^4x\mathcal{L}_0 + \int d^4x(\bar{R}^p_{qr}\eta^{qr}_p + \bar{\eta}^i_{jk}R^{jk}_i)\right] \\ &\quad\times(\bar{R}^p_{qr}R^{jk}_i(\delta^n_p(\delta^q_l\delta^r_m - \delta^q_m\delta^r_l)\delta^i_j(\delta^a_j\delta^b_k - \delta^a_k\delta^b_j)) \end{aligned} \quad (3.31)$$

From QFT (see equation (3.24)),

$$\begin{aligned}\int \mathcal{D}R\,\mathcal{D}\bar{R}\,\exp[-\,\mathrm{i}\int d^4x\mathcal{L}_0 + \int d^4x(\bar{R}^p_{qr}\eta^{qr}_p + \bar{\eta}^i_{jk}R^{jk}_i)](\bar{R}^p_{qr}R^{jk}_i) \\ = \frac{1}{\not{p} - M + i\epsilon}\delta^p_i(\delta^j_q\delta^k_r - \delta^j_r\delta^k_q)\end{aligned} \quad (3.32)$$

Then, by replacing the above expression in equation (3.31) we obtain

$$\begin{aligned}[R^{ab}_c\bar{R}^n_{lm}] &= \frac{4}{2}(\frac{1}{\not{p} - M + i\epsilon})(\delta^n_c\delta^a_l\delta^b_m - \delta^n_c\delta^a_m\delta^b_l) \\ &= 2\frac{1}{\not{p} - M + i\epsilon}\delta^n_c(\delta^a_l\delta^b_m - \delta^a_m\delta^b_l).\end{aligned} \quad (3.33)$$

The effective operator for $50^{(r)}$ becomes

$$[R^{ab}_c \bar{R}^n_{lm}] \rightarrow -2\frac{1}{M}\delta^n_c(\delta^a_l\delta^b_m - \delta^a_m\delta^b_l) \tag{3.34}$$

The $50^{(r)}$ can be reduced to $5\oplus 45$. Where $5^a = R^{ja}_j$. Then, we can extract the part of $50^{(r)}$ corresponding to 5 to guarantee $45^{ab}_a = 0$. So,

$$45^n_{lm} = 50^n_{lm} - (50^b_{bm}\delta^n_l + 50^b_{lb}\delta^n_m). \tag{3.35}$$

The final result is the whole effective propagator for the 45 representation:

$$[45^{ab}_c\bar{45}^n_{lm}] \rightarrow \frac{1}{2M}\{\delta^a_c(\delta^b_l\delta^n_m - \delta^b_m\delta^n_l) - \delta^b_c(\delta^a_l\delta^n_m - \delta^a_m\delta^n_l) + 4\delta^n_c(\delta^a_l\delta^b_m - \delta^a_m\delta^b_l)\} \tag{3.36}$$

3.4.2 Some effective operators

Here we present some effective operators that will be useful to the specific coefficients of the mass matrices.

$$\begin{aligned}
5^a\bar{5}_b &\rightarrow \frac{1}{M}\delta^a_b. && (3.37)\\
10^{ab}\bar{10}_{lm} &\rightarrow \frac{1}{M}(\delta^a_l\delta^b_m - \delta^a_m\delta^b_l). && (3.38)\\
15^{ab}\bar{15}_{lm} &\rightarrow \frac{1}{M}(\delta^a_l\delta^b_m + \delta^a_m\delta^b_l). && (3.39)\\
25^{(r)ab}\bar{25}^{(r)}_{lm} &\rightarrow \frac{2}{M}\delta^a_l\delta^b_m. && (3.40)\\
45^{ab}_c\bar{45}^n_{lm} &\rightarrow -\frac{1}{2M}\{\delta^a_c(\delta^b_l\delta^n_m - \delta^b_m\delta^n_l) - \delta^b_c(\delta^a_l\delta^n_m - \delta^a_m\delta^n_l) + \\
&\quad +4\delta^n_c(\delta^a_l\delta^b_m - \delta^a_m\delta^b_l)\}. && (3.41)\\
40^{abc}\bar{40}^{lmn} &\rightarrow \frac{3}{M}\delta^c_n(\delta^a_l\delta^b_m - \delta^a_m\delta^b_l) - \frac{1}{2M}\epsilon^{ijabc}\epsilon_{ijlmn} && (3.42)\\
50^{(r)ab}_c\bar{50}^{(r)n}_{lm} &\rightarrow -\frac{2}{M}\delta^n_c(\delta^a_l\delta^b_m - \delta^a_m\delta^b_l). && (3.43)\\
70^{ab}_c\bar{70}^n_{lm} &\rightarrow -\frac{1}{2}\{\delta^a_c(\delta^b_l\delta^n_m + \delta^b_m\delta^n_l) + \delta^b_c(\delta^a_l\delta^n_m + \delta^a_m\delta^n_l) \\
&\quad -6\delta^n_c(\delta^a_l\delta^b_m + \delta^a_m\delta^b_l))\}. && (3.44)
\end{aligned}$$

3.4.3 Explicit calculation in an anomalous model

There is a charge assignment that satisfies (3.8) and (3.9) but not (3.4). This kind of model is called *anomalous*. The anomaly can be cacelled by using the *Green-Schwarz mechanism* [Green and Schwarz 1984]. In this section, we shall compute the specific coefficients for the matrices built with the charges shown in table 3.4. The resulting matrices are given by (3.45).

f_1	f_2	f_3	t_1	t_2	t_3	f_u
-4	-2	2	5/3	2/3	-1/3	2/3

Table 3.4: Anomalous charges assignment for fermions in $SU(5)\times U(1)$

$$\mathcal{M}^u \sim \begin{pmatrix} \epsilon^4 & \epsilon^3 & \epsilon^2 \\ \epsilon^3 & \epsilon^2 & \epsilon^1 \\ \epsilon^2 & \epsilon^1 & 1 \end{pmatrix} \qquad \mathcal{M}^d \sim \begin{pmatrix} \epsilon^3 & \epsilon^4 & \epsilon^5 \\ \epsilon^1 & \epsilon^2 & \epsilon^3 \\ \epsilon^3 & \epsilon^2 & \epsilon^1 \end{pmatrix} \tag{3.45}$$

It can be noted that $\mathcal{M}^d$ has also problems in the entry 21. This implies that we must forbid some representations in order to reduce the order of magnitude down to ϵ^3. Using the same procedure as above, we find that the forbidden representations are:

$$15_{8/3}, \qquad 5_1, \qquad 45_1, \qquad 10_{8/3}, \qquad 15_{5/3}, \qquad 45_2 \tag{3.46}$$

The $\mathcal{M}^d_{33}$ entry

We start our specific calculation with the lowest order in the matrix for down-type quarks. There are two possible Lagrangian for this entry, namely

$$\mathcal{L} = \bar{5}\Sigma R + \bar{R}\bar{5}^{\phi_d}10 \tag{3.47}$$

$$\mathcal{L} = \bar{5}\bar{5}^{\phi_d}R + \bar{R}\Sigma 10 \tag{3.48}$$

For the Lagrangian (3.47) we have that the possible representations for R are 5 and 45. To find the representations, we use the same arguments as for (3.12) and the table 3.3. We can compose a $50^{(r)}$ representation with those representations. Then, the Lagrangian can be written as

$$\mathcal{L}^{50} = \bar{5}_a\Sigma^c_b 50^{ab}_c + \bar{50}^n_{lm}\bar{5}^{\phi_d}_n 10^{lm} \tag{3.49}$$

The following step is to introduce the effective propagator (3.43) and use the Σ VEV $\langle\Sigma^a_b\rangle = \langle V_a\rangle\delta^a_b$, given in (1.33).

$$\begin{aligned}
\mathcal{L}^{50}_{\rm eff} &= \bar{5}_a\Sigma^c_b[50^{ab}_c\bar{50}^n_{lm}]\bar{5}^{\phi_d}_n 10^{lm} \\
&= -\frac{2}{M}\bar{5}_a\Sigma^c_b(\delta^n_c(\delta^a_l\delta^b_m - \delta^a_m\delta^b_l))\bar{5}^{\phi_d}_n 10^{lm} \\
&= -\frac{2}{M}\bar{5}_a V_c\delta^c_b(\delta^n_c(\delta^a_l\delta^b_m - \delta^a_m\delta^b_l))\bar{5}^{\phi_d}_n 10^{lm} \\
&= -\frac{2}{M}\bar{5}_a V_5(\delta^a_l\delta^5_m - \delta^a_m\delta^5_l))\langle\bar{5}^{\phi_d}\rangle 10^{lm} \\
&= -\frac{2}{M}(-3V)(\bar{5}_a\langle\bar{5}^{\phi_d}\rangle 10^{a5} - \bar{5}_a\langle\bar{5}^{\phi_d}\rangle 10^{5a}) \\
&= \frac{12V}{M}(\bar{5}_a\langle\bar{5}^{\phi_d}\rangle 10^{a5}) \\
&= 12\frac{V}{M}\langle\bar{5}^{\phi_d}\rangle(b^c_i b_i + \tau^c\tau)
\end{aligned} \tag{3.50}$$

Where $i=1,2,3$.

Now, for the Lagrangian (3.48) the possible representations for R are 10 and 15. So, we use a $25^{(r)}$.

$$\mathcal{L}^{25}=\bar{5}_a\bar{5}^{\phi_d}_b R^{ab}+\bar{R}_{lm}\Sigma^m_n 10^{ln} \tag{3.51}$$

The effective Lagrangian becomes

$$\begin{aligned}\mathcal{L}^{25} &= \bar{5}_a\bar{5}^{\phi_d}_b[R^{ab}\bar{R}_{lm}]\Sigma^m_n 10^{ln}\\ &= \bar{5}_a\bar{5}^{\phi_d}_b(\frac{2}{M}\delta^a_l\delta^b_m)\Sigma^m_n 10^{ln}\\ &= \frac{2}{M}\bar{5}_l\langle\bar{5}^{\phi_d}\rangle\delta^5_b V_b\delta^b_n 10^{ln}\\ &= \frac{2}{M}\bar{5}_l\langle\bar{5}^{\phi_d}\rangle(-3V)10^{l5}\\ &= -6\frac{V}{M}\bar{5}_l\langle\bar{5}^{\phi_d}\rangle 10^{l5}\\ &= -6\frac{V}{M}\langle\bar{5}^{\phi_d}\rangle(b^c_i b_i+\tau^c\tau)\end{aligned} \tag{3.52}$$

Then, from (3.50) and (3.52) the coefficient for the 33 entry of the down-matrix is

$$O(\epsilon)\rightarrow 12-6=6 \tag{3.53}$$

The $\mathcal{M}^d_{22}$ entry

The entry 22 of the down matrix has order 2. This implies that we have to introduce two heavy fields R and F. There are three possible Lagrangian densities.

$$\mathcal{L} = \bar{5}\,\bar{5}^{\phi_d}\,R+\bar{R}\,\Sigma\,F+\bar{F}\,\Sigma\,10 \tag{3.54}$$
$$\mathcal{L} = \bar{5}\,\Sigma\,R+\bar{R}\,\bar{5}^{\phi_d}\,F+\bar{F}\,\Sigma\,10 \tag{3.55}$$
$$\mathcal{L} = \bar{5}\,\Sigma\,R+\bar{R}\,\Sigma\,F+\bar{F}\,\bar{5}^{\phi_d}\,10 \tag{3.56}$$

When we take into account the forbidden representation constraints we obtain that only two effective operators contribute to the coefficient:

$$\begin{aligned}\mathcal{L}_{\text{eff}\,1} &= \bar{5}\,\Sigma\,[70]\,\Sigma\,[5]\,\bar{5}^{\phi_d}\,10\\ \mathcal{L}_{\text{eff}\,2} &= \bar{5}\,\Sigma\,[70]\,\Sigma\,[45]\,\bar{5}^{\phi_d}\,10\end{aligned} \tag{3.57}$$

We shall compute here the $\mathcal{L}_{\text{eff}\,1}$. The Lagrangian is written as

$$\mathcal{L}=\bar{5}_a\Sigma^c_b 70^{ab}_c+\bar{70}^n_{lm}\Sigma^l_{\ n}5^m+\bar{5}_p\bar{5}^{\phi_d}_q 10^{pq} \tag{3.58}$$

The next step is to include the effective operators for the 5 and 70 representations.

$$
\begin{aligned}
\mathcal{L}_{\text{eff }1} &= \bar{5}_a\Sigma^c_b[\frac{6}{M}\delta^n_c(\delta^a_l\delta^b_m - \delta^a_m\delta^b_l) - \frac{1}{M}(\delta^a_c(\delta^b_l\delta^n_m + \delta^b_m\delta^n_l) + \delta^b_c(\delta^a_l\delta^n_m + \delta^a_m\delta^n_l)] \\
&\quad \times\Sigma^l_n(\frac{1}{M}\delta^m_p)\bar{5}^{\phi_d}_q 10^{pq} \\
&= \frac{1}{M^2}(6\,\bar{5}_l\langle\Sigma_n\rangle\delta^n_m\langle\Sigma_l\rangle\delta^l_n\bar{5}^{\phi_d}_l 10^{mq} + 6\,\bar{5}_m\langle\Sigma_n\rangle\delta^n_l\langle\Sigma_l\rangle\delta^l_n\bar{5}^{\phi_d}_q 10^{mq} \\
&\quad -\bar{5}_m\langle\Sigma_c\rangle\delta^c_l\langle\Sigma_l\rangle\delta^l_m\bar{5}^{\phi_d}_q 10^{mq} \\
&= \frac{V^2}{M^2}\langle\bar{5}^{\phi_d}\rangle(200\,\bar{5}_i\,10^{i5} + 225\,\bar{5}_4\,10^{45}) \qquad (3.59)
\end{aligned}
$$

Using the same procedure, the contribution for leptons and quarks, from the second Lagrangian, are respectively 225 and -200.

The $\mathcal{M}^u_{23}$ entry

The entry $\mathcal{M}^u_{33} \sim \epsilon$ involves two possible diagrams as shown in figure 3.1. This implies four effective operators:

$$
\begin{aligned}
\mathcal{L}_{\text{eff }1} &= 10\Sigma[10]5^{\phi_u}10 \\
\mathcal{L}_{\text{eff }2} &= 10\Sigma[\bar{40}]5^{\phi_u}10 \\
\mathcal{L}_{\text{eff }3} &= 105^{\phi_u}[10]\Sigma 10 \\
\mathcal{L}_{\text{eff }4} &= 105^{\phi_u}[\bar{40}]\Sigma 10 \qquad (3.60)
\end{aligned}
$$

We shall write the calculation for $\mathcal{L}_{eff\,4}$. The rest of results are shown in table 3.13.

We have that the exact Lagrangian is given

$$
\mathcal{L} = 10^{lm}5^{\phi_u\,n}\bar{40}_{lmn} + 40^{abe}\Sigma^d_f 10^{ef}\epsilon_{abcde} \qquad (3.61)
$$

Then, we introduce the effective propagator (3.42).

$$
\begin{aligned}
\mathcal{L}_{\text{eff}} &= 10^{lm}5^{\phi_u\,n}[\frac{3}{M}\delta^n_c(\delta^a_l\delta^b_m - \delta^a_m\delta^b_l) - \frac{1}{2M}\epsilon^{ijabc}\epsilon_{ijlmn}]\Sigma^d_f 10^{ef} \\
&= \frac{6}{M}10^{ab}5^{\phi_u\,c}\Sigma^d_f 10^{ef}\epsilon_{abcde} \\
&\quad -\frac{1}{2M}10^{lm}5^{\phi_u\,n}\Sigma^d_f\epsilon^{ijabc}\epsilon_{ijlmn}\epsilon_{abcde} \\
&= \frac{6}{M}10^{ab}5^{\phi_u\,c}\Sigma^d_f 10^{ef}\epsilon_{abcde} \\
&\quad -\frac{1}{2M}10^{lm}5^{\phi_u\,n}\Sigma^d_f(\delta^i_d\delta^j_e - \delta^i_e\delta^j_d)\epsilon_{ijlmn} \\
&= \frac{6}{M}10^{ab}5^{\phi_u\,c}\Sigma^d_f 10^{ef}\epsilon_{abcde} - \frac{1}{M}10^{lm}5^{\phi_u\,n}\Sigma^d_f\epsilon_{delmn} \\
&= \frac{5}{M}10^{ab}5^{\phi_u\,c}\Sigma^d_f 10^{ef}\epsilon_{abcde}
\end{aligned}
$$

Let us take $10^{41}10^{23}$, $\langle\Sigma^a_b\rangle = \langle V_a\rangle\delta^a_b$ and $\langle 5^{\phi_u\,c}\rangle = \langle 5^{\phi_u}\rangle\delta^c_5$.

$$\begin{aligned}\mathcal{L}_{\text{eff}} &= \frac{2\times 5}{M}\langle 5^{\phi_u}\rangle(10^{41}10^{23}\langle V_d\rangle\delta^d_3\epsilon_{d2415} + 10^{41}10^{32}\langle V_d\rangle\delta^d_2\epsilon_{d3415}\\ &\quad +10^{32}10^{41}\langle V_d\rangle\delta^d_1\epsilon_{d4325} + 10^{32}10^{14}\langle V_d\rangle\delta^d_4\epsilon_{d1325})\\ &= 30\frac{V}{M}\langle 5^{\phi_u}\rangle 10^{41}10^{23}\end{aligned} \tag{3.62}$$

Then, the contribution from this effective operator is 30.

3.5 Results for fermion matrices

We present the coefficients obtained for some entries of up, down and charged lepton mass matrices.

The contributions from allowed effective operators for each entry in the matrices for down quarks and leptons are presented in tables 3.5 to 3.12.

For the entry 22 at order ϵ^2 (table 3.9), the quark coefficient vanishes, then we must look for contributions at order ϵ^3. These contributions are given in table 3.10. These coefficients and those involving the 40 representation have been taken from [Nardi et al 2007]. We have redefined ϵ as $\epsilon/\sqrt{60}$ because of the normalization factor of Σ . The arrows $\uparrow$ and $\downarrow$ indicate the calculation derived form the two different possible contractions of the representation.

We also present some partial results that we have obtained with the up type quark matrix. This calculation is a work in progress,we show, in tables 3.11 to 3.15, some of the contributions computed up to now.

The matrix obtained for down quarks is given by

$$\mathcal{M}'^d \sim \begin{pmatrix} 1110\epsilon^3 & 5139\epsilon^4 & 106007\epsilon^5 \\ -1070\epsilon^3 & -1000\epsilon^3 & -2050\epsilon^3 \\ -299\epsilon^3 & 232\epsilon^2 & 6\epsilon^1 \end{pmatrix} \tag{3.63}$$

We have used a random number for entries 12 and 13 since they are highly suppressed. The matrix $\mathcal{D}^\dagger_L$ (see equation (2.5)) obtained for such a matrix is

$$\mathcal{D}^\dagger_L = \begin{pmatrix} 0.98500 & -0.17249 & 0.00191 \\ -0.17249 & -0.98501 & 0.00138 \\ -0.00164 & 0.00169 & 0.99999 \end{pmatrix} \tag{3.64}$$

And the diagonal matrix for quark masses is

$$\mathcal{M}^d = \begin{pmatrix} 0.00479 & 0 & 0 \\ 0 & 0.13315 & 0 \\ 0 & 0 & 5.81271 \end{pmatrix} \tag{3.65}$$

The mass ratios obtained are

$$\frac{m_d}{m_s} = 0.0359 \qquad \frac{m_s}{m_b} = 0.0229 \tag{3.66}$$

Effective operator	Lepton	Down
$\bar{5}\Sigma[50^{(r)}]\bar{5}^{\phi_d}10$	12	12
$\bar{5}\bar{5}^{\phi_d}[25^{(r)}]\Sigma 10$	-6	-6
Coefficient	6	6

Table 3.5: Contributions for $\mathcal{M}^d_{33} \sim \epsilon$ for leptons and down-type quarks

Effective operator	Lepton	Down
$\bar{5}\bar{5}^{\phi_d}[10]\Sigma[10]\Sigma 10$	36	1
$\bar{5}\bar{5}^{\phi_d}[10]\Sigma[15]\Sigma 10$	0	-25
$\bar{5}\bar{5}^{\phi_d}[10]\Sigma[\bar{40}]\Sigma 10$	0	50
$\bar{5}\bar{5}^{\phi_d}[15]\Sigma[10]\Sigma 10$	0	5
$\bar{5}\bar{5}^{\phi_d}[15]\Sigma[15]\Sigma 10$	0	-5
$\bar{5}\Sigma[5]\bar{5}^{\phi_d}[10]\Sigma 10$	-18	2
$\bar{5}\Sigma[5]\bar{5}^{\phi_d}[15]\Sigma 10$	0	10
$\bar{5}\Sigma[45]\bar{5}^{\phi_d}[10]\Sigma 10$	90	10
$\bar{5}\Sigma[45]\bar{5}^{\phi_d}[\bar{40}]\Sigma 10$	0	200
$\bar{5}\Sigma[70]\bar{5}^{\phi_d}[15]\Sigma 10$	0	-100
$\bar{5}\Sigma[5]\Sigma[5]\bar{5}^{\phi_d}10$	9	4
$\bar{5}\Sigma[5]\Sigma[45]\bar{5}^{\phi_d}10$	-45	20
$\bar{5}\Sigma[45]\Sigma[5]\bar{5}^{\phi_d}10$	75	100
$\bar{5}\Sigma[45\uparrow]\Sigma[45]\bar{5}^{\phi_d}10$	390	-60
$\bar{5}\Sigma[45\downarrow]\Sigma[45]\bar{5}^{\phi_d}10$	105	20
$\bar{5}\Sigma[70]\Sigma[5]\bar{5}^{\phi_d}10$	225	200
$\bar{5}\Sigma[70]\Sigma[45]\bar{5}^{\phi_d}10$	225	-200
Coefficient	1092	232

Table 3.6: Contributions for $\mathcal{M}^d_{32} \sim \epsilon^2$ for leptons and down-type quarks

We can observe that ratios from equation (3.66) fit approximately those found in (3.1). In addition, the entry 12 of $\mathcal{D}_L^\dagger$ is below the experimental value of $\theta_{12} \approx 0.22$ and the entry 13 is less than the value of $\theta_{12} \approx 0.003$. This means that the model is realistic and we can consider this matrix to be the CKM one with little corrections provided by the matrix $\mathcal{U}_L$ once the coefficients are obtained, but this will be undertaken in an ulterior work.

Effective operator	Lepton	Down
$\bar{5}\bar{5}^{\phi_d}[10]\Sigma[10]\Sigma[10]\Sigma 10$	0	1
$\bar{5}\bar{5}^{\phi_d}[10]\Sigma[10]\Sigma[15]\Sigma 10$	0	-25
$\bar{5}\bar{5}^{\phi_d}[25^{(r)}]\Sigma[10]\Sigma[\bar{40}]\Sigma 10$	0	-300
$\bar{5}\bar{5}^{\phi_d}[10]\Sigma[15]\Sigma[10]\Sigma 10$	0	-25
$\bar{5}\bar{5}^{\phi_d}[10]\Sigma[15]\Sigma[15]\Sigma 10$	0	25
$\bar{5}\bar{5}^{\phi_d}[10]\Sigma[\bar{40}]\Sigma[10]\Sigma 10$	-54	-5
$\bar{5}\bar{5}^{\phi_d}[10]\Sigma[\bar{40}\uparrow]\Sigma[\bar{40}]\Sigma 10$	0	-200
$\bar{5}\bar{5}^{\phi_d}[10]\Sigma[\bar{40}\downarrow]\Sigma[\bar{40}]\Sigma 10$	0	350
$\bar{5}\bar{5}^{\phi_d}[15]\Sigma[10]\Sigma[10]\Sigma 10$	0	5
$\bar{5}\bar{5}^{\phi_d}[15]\Sigma[10]\Sigma[15]\Sigma 10$	0	-125
$\bar{5}\bar{5}^{\phi_d}[15]\Sigma[15]\Sigma[10]\Sigma 10$	0	-5
$\bar{5}\bar{5}^{\phi_d}[15]\Sigma[15]\Sigma[15]\Sigma 10$	0	5
Coefficient	-54	-299

Table 3.7: Contributions for $\mathcal{M}^d_{31} \sim \epsilon^3$ for charged leptons and down type quarks

Effective operator	Lepton	Down
$\bar{5}\Sigma_1[70]\Sigma_1[5]\bar{5}^{\phi_d}[10]\Sigma_{-1}10$	1350	200
$\bar{5}\Sigma_1[70]\Sigma_1[5]\bar{5}^{\phi_d}[15]\Sigma_{-1}10$	0	1000
$\bar{5}\Sigma_1[70]\Sigma_1[45]\bar{5}^{\phi_d}[10]\Sigma_{-1}10$	1350	-200
$\bar{5}\Sigma_1[70]\Sigma_1[45]\bar{5}^{\phi_d}[\bar{40}]\Sigma_{-1}10$	0	-4000
$\bar{5}\Sigma_1[70]\Sigma_1[70]\bar{5}^{\phi_d}[15]\Sigma_{-1}10$	0	-400
$\bar{5}\Sigma_1[70]\Sigma_1[70]\bar{5}^{\phi_d}[15]\Sigma_{-1}10$	0	800
$\bar{5}\Sigma_{-1}[5]\bar{5}^{\phi_d}[10]\Sigma_1[\bar{40}]\Sigma_1 10$	0	-100
$\bar{5}\Sigma_{-1}[45]\bar{5}^{\phi_d}[10]\Sigma_1[\bar{40}]\Sigma_1 10$	0	1200
$\bar{5}\Sigma_{-1}[45\uparrow]\bar{5}^{\phi_d}[\bar{40}]\Sigma_1[\bar{40}]\Sigma_1 10$	-1248	1044
$\bar{5}\Sigma_{-1}[45\downarrow]\bar{5}^{\phi_d}[\bar{40}]\Sigma_1[\bar{40}]\Sigma_1 10$	456	-614
Coefficient	1908	-1070

Table 3.8: Contributions for $\mathcal{M}^d_{21} \sim \epsilon^3$ for leptons and down type quarks

Effective operator	Lepton	Down
$\bar{5}\Sigma[70]\Sigma[5]\bar{5}^{\phi_d}10$	225	200
$\bar{5}\Sigma[70]\Sigma[45]\bar{5}^{\phi_d}10$	225	-200
Coefficient	500	0

Table 3.9: Contributions for $\mathcal{M}^d_{22} \sim \epsilon^2$ for leptons and down type quarks

Effective operator	Lepton	Down
$\bar{5}\Sigma_1[70\uparrow]\Sigma_0[70]\Sigma_1[45]\bar{5}^{\phi_d}10$	-4725	-800
$\bar{5}\Sigma_1[70\downarrow]\Sigma_0[70]\Sigma_1[45]\bar{5}^{\phi_d}10$	-675	1600
$\bar{5}\Sigma_0[70\uparrow]\Sigma_1[70]\Sigma_1[45]\bar{5}^{\phi_d}10$	-4725	-800
$\bar{5}\Sigma_0[70\downarrow]\Sigma_1[70]\Sigma_1[45]\bar{5}^{\phi_d}10$	-675	1600
$\bar{5}\Sigma_0[70\uparrow]\Sigma_1[70]\bar{5}^{\phi_d}[15]\Sigma_1 10$	0	-400
$\bar{5}\Sigma_0[70\downarrow]\Sigma_1[70]\bar{5}^{\phi_d}[15]\Sigma_1 10$	0	800
$\bar{5}\Sigma_0[70]\Sigma_1[5]\bar{5}^{\phi_d}[10]\Sigma_1 10$	1350	200
$\bar{5}\Sigma_0[70]\Sigma_1[5]\bar{5}^{\phi_d}[15]\Sigma_1 10$	0	1000
$\bar{5}\Sigma_0[70]\Sigma_1[45]\bar{5}^{\phi_d}[10]\Sigma_1 10$	1350	-200
$\bar{5}\Sigma_0[70]\Sigma_1[45]\bar{5}^{\phi_d}[40]\Sigma_1 10$	0	-4000
Coefficient	-8100	-1000

Table 3.10: Contributions for $\mathcal{M}^d_{22} \sim \epsilon^3$ for leptons and down type quarks

Effective operator	Lepton	Down
$\bar{5}\Sigma[70]\Sigma[5]\bar{5}^{\phi_d}[10]\Sigma 10$	1350	200
$\bar{5}\Sigma[70]\Sigma[5]\bar{5}^{\phi_d}[15]\Sigma 10$	0	1000
$\bar{5}\Sigma[70]\Sigma[45]\bar{5}^{\phi_d}[10]\Sigma 10$	1350	-200
$\bar{5}\Sigma[70]\Sigma[45]\bar{5}^{\phi_d}[15]\Sigma 10$	0	-50
$\bar{5}\Sigma[70]\Sigma[5]\Sigma[5]\bar{5}^{\phi_d}10$	-675	400
$\bar{5}\Sigma[70]\Sigma[5]\Sigma[45]\bar{5}^{\phi_d}10$	1125	1000
$\bar{5}\Sigma[70]\Sigma[45]\Sigma[5]\bar{5}^{\phi_d}10$	1125	-2000
$\bar{5}\Sigma[70]\Sigma[45\uparrow]\Sigma[45]\bar{5}^{\phi_d}10$	4275	1200
$\bar{5}\Sigma[70]\Sigma[45\downarrow]\Sigma[45]\bar{5}^{\phi_d}10$	1575	-400
$\bar{5}\Sigma[70\uparrow]\Sigma[70]\Sigma[5]\bar{5}^{\phi_d}10$	-4725	800
$\bar{5}\Sigma[70\downarrow]\Sigma[70]\Sigma[5]\bar{5}^{\phi_d}10$	-675	-1600
$\bar{5}\Sigma[70\uparrow]\Sigma[70]\Sigma[45]\bar{5}^{\phi_d}10$	-4725	-800
$\bar{5}\Sigma[70\downarrow]\Sigma[70]\Sigma[45]\bar{5}^{\phi_d}10$	-675	-1600
Coefficient	-675	-2050

Table 3.11: Contributions for $\mathcal{M}^d_{23} \sim \epsilon^3$ for leptons and down type quarks

Effective operator	Lepton	Down
$\bar{5}\Sigma[45]\bar{5}^{\phi_d}[10]\Sigma[\bar{40}]\Sigma 10$	0	500
$\bar{5}\Sigma[45\uparrow]\bar{5}^{\phi_d}[\bar{40}]\Sigma[\bar{40}]\Sigma 10$	0	-800
$\bar{5}\Sigma[45\downarrow]\bar{5}^{\phi_d}[\bar{40}]\Sigma[\bar{40}]\Sigma 10$	0	1400
$\bar{5}\bar{5}^{\phi_d}[25^{(r)}]\Sigma[10]\Sigma[\bar{40}]\Sigma 10$	945	10
Coefficient	945	1110

Table 3.12: Contributions for $\mathcal{M}^d_{11} \sim \epsilon^3$ for leptons and down type quarks

Effective operator	Up
$10\Sigma[10]5^{\phi_u}10$	-6
$10\Sigma[\bar{40}]5^{\phi_u}10$	-30
$105^{\phi_u}[10]\Sigma10$	12
$105^{\phi_u}[\bar{40}]\Sigma10$	30
Coefficient	6

Table 3.13: Contributions for $\mathcal{M}^u_{23} \sim \epsilon$ for up type quarks

Effective operator	Up
$10\Sigma[10]\Sigma[10]5^{\phi_u}10$	17
$10\Sigma[10]\Sigma[\bar{40}]5^{\phi_u}10$	-60
$10\Sigma[15]\Sigma[10]5^{\phi_u}10$	-25
$10\Sigma[\bar{40}]\Sigma[10]5^{\phi_u}10$	-60
$10\Sigma[\bar{40}]\Sigma[\bar{40}]5^{\phi_u}10$	510
$10\Sigma[10]5^{\phi_u}[10]\Sigma10$	-8

Table 3.14: Contributions for $\mathcal{M}^u_{22} \sim \epsilon^2$ for up type quarks

Effective operator	Up
$10\,5^{\phi_u}[10]\Sigma[10]\Sigma10$	12
$10\,5^{\phi_u}[10]\Sigma[15]\Sigma10$	-25
$10\Sigma[10]5^{\phi_u}[10]\Sigma10$	-8
$10\Sigma[10]\Sigma[10]5^{\phi_u}10$	17
$10\Sigma[10]\Sigma[\bar{40}]5^{\phi_u}10$	-60
$10\Sigma[15]\Sigma[10]5^{\phi_u}10$	-25
$10\Sigma[\bar{40}]\Sigma[10]5^{\phi_u}10$	-60
$10\Sigma[\bar{40}]\Sigma[\bar{40}]5^{\phi_u}10$	-510

Table 3.15: Contributions for $\mathcal{M}^u_{31} \sim \epsilon^2$ for up type quarks

Conclusions and open issues

We studied the fundamentals of a Grand Unified Theory $SU(5)$. In particular, we have seen the necessity of introducing the SUSY extension of the model in order to achieve the gauge coupling unification and to be consistent with experimental upper limits for proton decay lifetime.

One of the problems that is non solved by minimal SUSY $SU(5)$ is the fermion mass hierarchy problem in which we have a large structure of masses among the three fermion families. We made a study of this problem and the mechanism proposed by Froggatt and Nielsen. This mechanism allows predict the mass hierarchy, but not the specific coefficients, if the U_H symmetry breaking is induced by a SM singlet. We have implemented a modification to this mechanism using a $SU(5)$ adjoint representation to break the Abelian horizontal symmetry. In this way, we could get specific values for the Yukawa couplings. We have considered two models: *i*) A non-anomalous model in which we checked the possibility to obtain the mixing matrix using the hierarchy predicted by the $\mathcal{M}'_u$ matrix as the down one is almost diagonal. *ii*) A second model resulting from an anomalous H-charge assignment assignation of horizontal charges. We have computed specifically the $\mathcal{M}'_d$ entries and we have obtained $\mathcal{D}_L$ which is involved in the CKM matrix.

There are some results that we would like to remark. First of all, we learned how to obtain an effective GUT model that fits with experimental data. Second, we obtained a $b-\tau$ unification at the GUT scale which is in agreement with the expected data ([Ross-Serna 2007]). We also obtained the experimental relation for down quarks $m_d : m_s : m_b = \epsilon^3 : \epsilon^2 : \epsilon$ (for $\epsilon = 0.25$). Thus, the model we have considered can be regarded as a predictive and successful approach to the solution of the fermion mass hierarchy problem.

We want to stress that the results presented in this research work are preliminary and therefore there remain several open issues. The first one is related with the calculation of possible coefficients corresponding to up-type quarks to obtain a most consistent CKM matrix. Also, we have not told anything about neutrino masses, this topic should be considered in an ulterior work. Another open question is related with the achievement of the ratios $3\,m_s/m_\mu = 0.70$ and $m_d/3\,m_e = 0.82$ which are not very included in this work. Finally, a later work should study the Green-Schwarz mechanism [Green and Schwarz 1984] which has not been treated in this text.

Appendix A
Group theory

During the twentieth century the symmetries played a central role in theoretical physics. That is the kernel of particle physics. The theory of symmetries is the group theory. Because of that, we dedicate an appendix to some basis on Group theory. A complete treatise on Group theory can be found in [Georgi 1999], [Stancu 1996] and [Cheng and Li 1984].

A.1 Elements

A *group* G is a set of elements with a product $(\cdot)$ satisfying the properties:

i) Closure. If $a, b \in G$, then $c = a \cdot b$ is also in G.

ii) Associative. $a(bc) = (ab)c$.

iii) Identity. $\exists e \in G$ such that $ea = ae = a$, $\forall a \in G$.

iv) Inverse $\forall a \in G$, $\exists\, a^{-1} \in G$ such that $aa^{-1} = a^{-1}a = e$.

If the product is commutative, we say G is *Abelian*.

Given two groups $G = \{g_1, g_2, ...\}$ and $H = \{h_1, h_2, ...\}$, if the elements of G commute with those of H, the *direct product* $G \times H = \{g_i H_j\}$ is defined with the product law

$$(g_k h_l) \cdot (g_m h_n) = (g_k \cdot g_m)(g_h \cdot h_k). \tag{A.1}$$

The direct-product groups are pretty important in Particle Physics. As a direct example, we find that the electroweak group is $SU(2) \times U(1)$. When a group can not be written as a direct product group, it is called a *simple group*.

Another main element in Group theory is the group *Representation*. A representation is a realisation of the group using matrices as elements. This is, a representation D is defined

$$D : a \to D(a) \in GL(n) \tag{A.2}$$

where $ab = c \to D(a)D(b) = D(c)$. $D(a)$ is called *reducible* is it can be expressed in a block-diagonal form ($D(a) = D_1(a) \oplus D_2(a) \oplus ...$). Otherwise, the representation is called *irreducible*.

There is a particular kind of groups with important applications in physics. This is the set of *Lie Groups*. These are continuous groups represented by unitary operators. The product law of a Lie group is given by

$$a(x) \cdot a(y) = a(f(x, y)). \tag{A.3}$$

From this composition law, it is possible to work in the commutator of two elements of a Lie group. As a result, we can obtain the *Lie algebra* for generators. Then, each Lie group defines a Lie algebra and vice versa. This algebra is expressed as

$$[X_j, X_k] = \mathrm{i}C^l_{jk}X_l \tag{A.4}$$

where $C^l_{jk} = C^l_{kj}$ are called the structure constants.

A.1.1 $SU(n)$

One type of Lie groups is the Unitary groups ($U(N)$). These are characterized by having unitary representations of order N: $U^\dagger U = I$. A particular set of unitary groups is the set of special unitary groups ($SU(N)$). These have the additional condition $\det U = 1$. The number of generators for $SU(N)$ is $n^2 - 1$ and each element of its representation can be written as

$$U = \exp\{\mathrm{i}\sum_{a=1}^{n^2-1} \epsilon_a J_a\} \tag{A.5}$$

where ϵ_a's are real parameters.

SU(2)

One example of special unitary group is $SU(2)$. The basis for its 2 dimensional representation is the set of Pauli's matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.6}$$

If we define the generators as $J_i = \sigma_i/2$, the algebra of generators is

$$[J_j, J_k] = \mathrm{i}\epsilon_{jkl}J_l \tag{A.7}$$

One of the most important representations of $SU(2)$ is the Angular Momentum operator representation.

A.2 The tensor method and Young tableaux

Any vector ψ_i in C_n is mapped by an $SU(N)$ transformation as

$$\begin{aligned} \psi_i &\rightarrow \psi'_i = U_i^j \psi_j \\ \psi^i &\rightarrow \psi'^i = U^i_j \psi^j \end{aligned} \tag{A.8}$$

where $U_i^{*j} = U_i^j$ and $\psi_i^* = \psi^i$.

A second-rank tensor ψ^{ij}, for instance, transforms as

$$\psi'^{ij} = U_k^i U_l^j \psi^{kl}. \tag{A.9}$$

We can decompose each tensor (ψ^{ij}, for example) in a symmetric part and an antisymmetric one:

$$S^{ij} = \frac{1}{\sqrt{2}}(\psi^{ij} + \psi^{ji}), \qquad A^{ij} = \frac{1}{\sqrt{2}}(\psi^{ij} - \psi^{ji}). \tag{A.10}$$

An important task in group theory is finding irreducible representations for $SU(N)$. To do this, we can use a pictorial tool called *Young tableaux*. In these diagrams, symmetric tensors are represented by horizontal boxes, and antisymmetric ones by vertical boxes. For example, $\psi^{ij} = S^{ij} + A^{ij}$ can be represented as:

$$\begin{aligned} S^{ij} &= \begin{array}{|c|c|}\hline i & j \\ \hline\end{array} \\ A^{ij} &= \begin{array}{|c|}\hline i \\ \hline j \\ \hline\end{array} \end{aligned} \tag{A.11}$$

In a Young tableau, the number of boxes in the row does not increase from top to bottom. To find the dimension of a Young diagram there are two definitions: *hook length*(h_i) and *distance to the first box*(D_i). h_i is defined as the total number of boxes that a hook passes including the box i.

$$\begin{array}{|c|c|}\hline \cdot & \cdots \\ \hline \vdots \\ \cline{1-1}\end{array} \quad h_i = 3, \qquad \begin{array}{|c|c|}\hline & \cdot \\ \hline \\ \cline{1-1}\end{array} \quad h_i = 1 \tag{A.12}$$

D_i is defined as the number of steps going from the box in the upper left-hand corner of the diagram to the ith box with each step towards the right counted as $+1$ and each downward step as -1

$$\begin{array}{|r|c|c|}\hline 0 & 1 & 2 \\ \hline -1 & 0 \\ \cline{1-2}\end{array} \tag{A.13}$$

Then, the dimension of a $SU(N)$ irreducible representation associated with any Young diagram is

$$d = \prod_i \frac{N + D_i}{h_i} \tag{A.14}$$

Taking the product over all boxes. For the $SU(5)$ diagram $\begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}$ the dimension is

$$\frac{5-1}{1} \cdot \frac{5+0}{2} = 10 \tag{A.15}$$

As an specific example, we can group the 27 states of 3 quarks in a set of representations of $SU(3)$. This can be seen in table A.1.

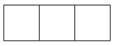
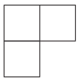
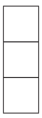
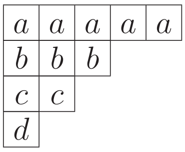
Tableau	ψ_{ijk}	Dimension
	u u u	
	u u u	**10**
	s s s	
	u u d	
	u u s	
	d d u	
	d d s	
	s s u	
	s s d	
	u d s	
	u u d	
	u u s	**8**
	d d u	
	d d s	
	s s u	
	s s d	
	u d s	**1**
		

Table A.1: Young tableaux for 3-quark states

Now, we shall see how to decompose a product of two representations of $SU(N)$:

(1) In the first factor, we must assign a letter (say a) to all boxes in the first row, another letter to the boxes in the second row, etc.

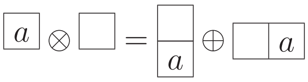


(A.16)

(2) Next, we attach boxes with a to the second diagram in all possible ways taking care of not putting two a's in the same column and the resultant is still a Young diagram. This process is repeated for b's, c's, etc.

(3) The added symbols are read from right to left in the first row, then in the second one, etc. This sequence of symbols ($aaabbaac$...) must form a sequence such that to the left of any symbol there are no fewer a than b, and no fewer b than c, etc.

Let us take an example in $SU(5)$



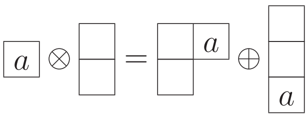
(A.17)

which corresponds to

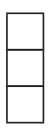
$$5 \otimes 5 = 10 \oplus 15 \tag{A.18}$$

Other examples:

i) $5 \otimes 10$:



(A.19)

For  the dimension is

$$d = \frac{3}{1}\frac{4}{2}\frac{5}{3} = 10, \tag{A.20}$$

and for 

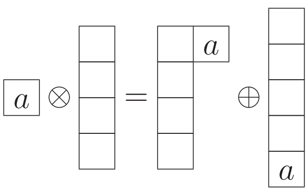
$$d = \frac{4}{1}\frac{5}{3}\frac{2}{1} = 40 \tag{A.21}$$

.

Then,

$$5 \otimes 10 = 40 \oplus \bar{10} \tag{A.22}$$

ii) $5 \otimes \bar{5}$:

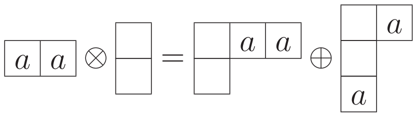


(A.23)

This means that

$$5 \otimes \bar{5} = 24 \oplus 1. \tag{A.24}$$

iii) $15 \otimes 10$:



(A.25)

Therefore,

$$15 \otimes 10 = 105 \oplus 45. \tag{A.26}$$

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