chapter 3 Vectors

Vector addition and component vectors. (Section 3.3)

Questions 1 and 2 may be used after Example 2 of Section 3.3, before the discussion of vectors in three dimensions.

1. Vector addition
   - The solution of this problem uses **i** and **j** notation, which may serve as a precursor to Fig. 3.17 in Section 3.3.
2. A trip
   - This problem involves addition of three vectors in two dimensions.

Vector multiplication: dot product. (Section 3.4, part 1)

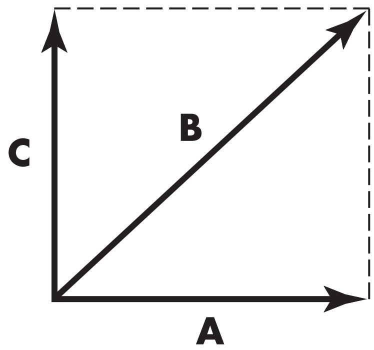
Problems 3–5 are to be used after Example 5 in Section 3.4.

3. The length of **A** + **B**
4. Law of cosine of a triangle
5. Determine the vector **R** OA using the dot product.

Vector multiplication: (Section 3.4, part 2)

Cross product. Problems 6 and 7 come after Example 6 in Section 3.4.

6. Area of a triangle
7. Law of sine of a triangle
8. Volume of a parallelepiped
   - This exercise involves the combined operations of one dot product and one cross product.

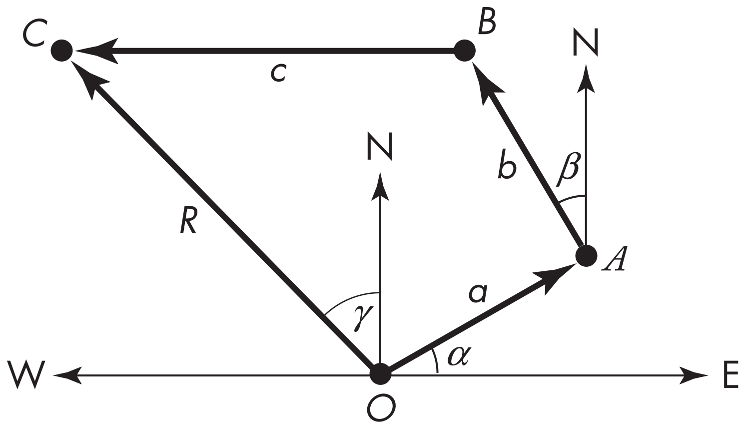


Given: $\mathbf{A} = \mathbf{i}$, $\mathbf{B} = \mathbf{i} + \mathbf{j}$, $\mathbf{C} = \mathbf{j}$. Define: $\mathbf{D} = \mathbf{A} + \mathbf{B} + \mathbf{C}$. Determine the magnitude of **D** and the angle θ_D that **D** makes with the *x* axis:

	1	2	3	4
D	2	$2\sqrt{2}$	2	$2\sqrt{2}$
θ_D	45°	45°	>45°	>45°

Hint: $D_x = A_x + B_x + C_x$, $D_y = A_y + B_y + C_y$.

Explanation: Separately adding the *x* and *y* components of **A, B,** and **C** yields $\mathbf{D} = 2\mathbf{i} + 2\mathbf{j}$. So $D = \sqrt{2^2 + 2^2} = 2\sqrt{2}$. $\tan \theta_D = D_y/D_x = 2/2 = 1$, or $\theta = 45°$. Answer = B.

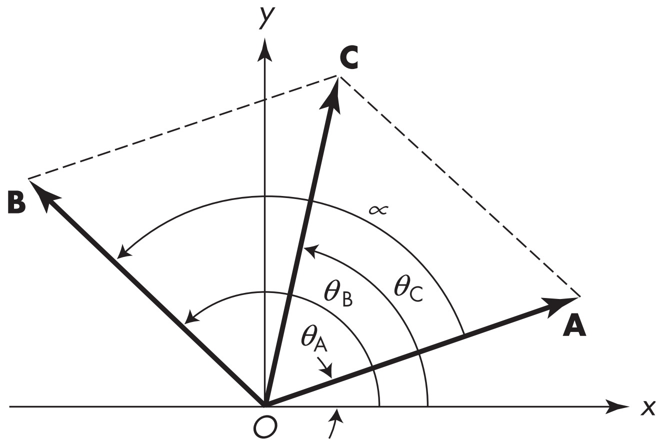


Consider a trip that goes from O to C by first going from O to A, then from A to B, and finally from B to C, as described by the vector diagram. The resultant displacement vector is **R** = ***OA*** + ***AB*** + ***BC,*** where the vector **BC** is parallel to the x axis.

The x component of the vector **R** is given by which of the following?

	R_x
A	$a \cos \alpha - b \sin \beta - c$
B	$a \cos \alpha + b \cos \beta + c$
C	$a \cos \alpha + b \cos \beta - c$

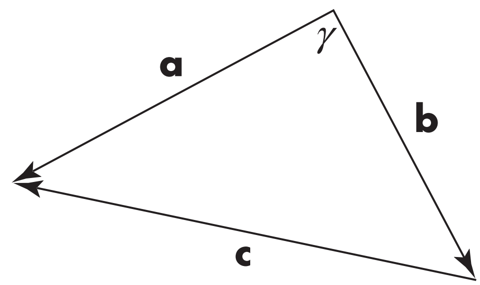
Explanation: From inspection of the Cartesian components of **A, B,** and **C** as shown in the vector diagram, the x components of **A** and **OA** is positive, and that of **AB** and **BC** are negative. Answer = A.



In the figure shown, **C** = **A** + **B**. Using the definition of the scalar product of two vectors, find the square of the length of **C**. Let the angle between **A** and **B** be α, where $\alpha = \theta_{\mathbf{B}} - \theta_{\mathbf{A}}$. Here are your choices:

A	$C^2 = A^2 + B^2 - 2AB \cos \theta_A$
B	$C^2 = A^2 + B^2 - 2AB \cos \theta_B$
C	$C^2 = OA^2 + B^2 - 2AB \cos \alpha$
D	$C^2 = A^2 + B^2 + 2AB \cos \alpha$

Explanation: For any **V**, using the relationship $\mathbf{V}^2 = \mathbf{V}\cdot\mathbf{V}$, $|\mathbf{C}|^2 = |\mathbf{A} + \mathbf{B}|^2 = |\mathbf{A}|^2 + |\mathbf{B}|^2 + 2AB \cos \alpha$. Answer = D.

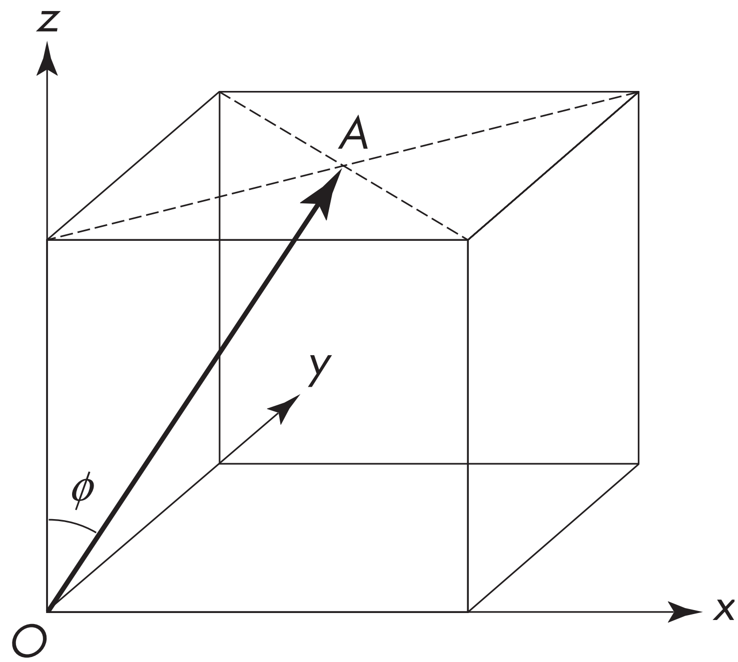


Consider the triangular setup that satisfies the vector relation $\mathbf{a} = \mathbf{b} + \mathbf{c}$. Denote the magnitudes of three sides by **a**, **b**, and **c**. Using the relation for any vector $\mathbf{V}$, $\mathbf{V}^2 = \mathbf{V}\cdot\mathbf{V}$, choose the correct expression for $\mathbf{c}^2$:

A	$a^2 + b^2 + 2ab$
B	$a^2 + b^2 - 2ab$
C	$a^2 + b^2 + 2ab \cos \gamma$
D	$a^2 + b^2 - 2ab \cos \gamma$

Explanation: From the vector relation given, $\mathbf{c} = \mathbf{a} - \mathbf{b}$. The dot product $\mathbf{c}\cdot\mathbf{c} = (\mathbf{a} - \mathbf{b})\cdot(\mathbf{a} - \mathbf{b}) = \mathbf{a}\cdot\mathbf{a} + \mathbf{b}\cdot\mathbf{b} - \mathbf{2a}\cdot\mathbf{b} = a^2 + b^2 - 2ab \cos \gamma$.
Answer = D.

 4. Law of Cosine of a Triangle

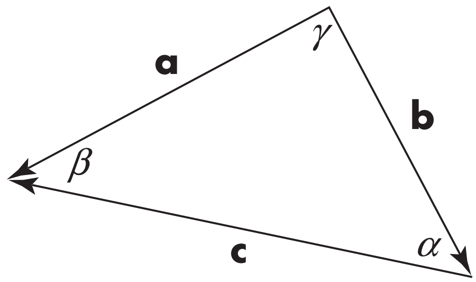


Consider the unit cube (each side is one unit) shown here. Using the coordinate system shown with the origin at *O*, the point *A*, which is at the center of the top surface, has coordinates (½, ½, 1). Using the dot product method, find R_{OA}, the distance between points *O* and *A*.

Extra: Find the angle ϕ formed between $\mathbf{R}_{OA}$ and the *z* axis.

Explanation: $\mathbf{R}_{OA} = \frac{1}{2}(\mathbf{i} + \mathbf{j}) + \mathbf{k}$. Using the relation for any vector **V**, $\mathbf{V}^2 = \mathbf{V}\cdot\mathbf{V}$, $R^2_{OA} = \frac{1}{4} + \frac{1}{4} + 1 = \frac{3}{2}$. So $R_{OA} = \sqrt{\frac{3}{2}}$.

Explanation—extra: From the diagram, $\cos\phi = \frac{1}{R_{OA}} = \sqrt{\frac{2}{3}}$, or $\phi \approx 35°$.



Consider the vectors **a, b,** and **c**, with **a** = **b** + **c**. They form a closed triangle. Use the cross product to determine the area of this triangle:

A	B	C	D
ab	$\frac{1}{2}$ ab	$\|\mathbf{a} \times \mathbf{b}\|$	$\frac{1}{2}\|\mathbf{a} \times \mathbf{b}\|$

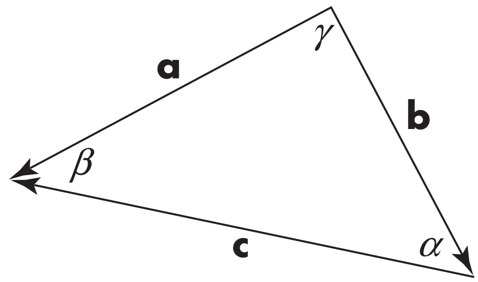
Hint: The area of a triangle is given by half of its base times its height.

Explanation: From the hint,

area $= \frac{1}{2} b\, h = \frac{1}{2}$ b a sin $\gamma = \frac{1}{2}\,|\mathbf{a} \times \mathbf{b}|$. Answer = D.

Notice that this area can be expressed using any of the pairs:

area $= \frac{1}{2}\,|\mathbf{a} \times \mathbf{b}| = \frac{1}{2}\,|\mathbf{b} \times \mathbf{c}| = \frac{1}{2}|\mathbf{c} \times \mathbf{a}|$.



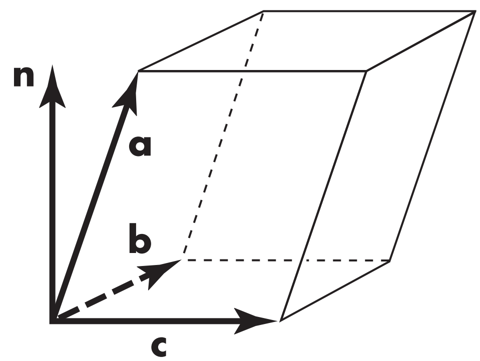
Given **b** + **c** = **a**, where the three vectors **a**, **b**, and **c** form a closed triangle as shown in the figure. Choose the correct relation:

A	$a \sin \alpha$
B	$\frac{a}{\sin \alpha} = \frac{c}{\sin \gamma}$
C	$ac = \sin \gamma$

Hint: From the previous problem, the area of the triangle is given by $\frac{1}{2}|\mathbf{a} \times \mathbf{b}| = \frac{1}{2}|\mathbf{b} \times \mathbf{c}|$.

Explanation: From the hint, $\frac{1}{2} ab \sin \gamma = \frac{1}{2} bc \sin \alpha \quad (1)$

Canceling the factors of $\frac{1}{2}$ and b on both sides of Eq. (1) and a rearrangement of terms leads to Answer B. From symmetry, we see that the area may also be written in as $\frac{1}{2} ac \sin \beta$, which together with Eq. (1) leads to $\frac{a}{\sin \alpha} = \frac{c}{\sin \gamma} = \frac{b}{\sin \beta}$. This set of relationships is referred to as the law of sines of a triangle.



Consider a parallelepiped formed by three vectors **a, b,** and **c**. The general expression for its volume is given by which of the following?

A	a b c
B	$\frac{1}{2}$ a b c
C	$\frac{1}{2}\ |\mathbf{a} \times \mathbf{b} \cdot \mathbf{c}|$
D	$|\mathbf{a} \times \mathbf{b} \cdot \mathbf{c}|$

Hint: The area *A* of the base (the parallelogram formed by vectors **b** and **c**) may be obtained using the cross product relation $\mathbf{b} \times \mathbf{c} = A\mathbf{n}$, where **n** is a unit vector perpendicular to the area (see the figure).

Explanation: The volume of the parallelepiped is given by
$V = h\ A = \mathrm{a} \cos \alpha\ A = \mathbf{a} \cdot \mathbf{n}\ A = |\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}|$, where $h = \mathbf{a} \cos \alpha$ was used.