chapter 13 Dynamics of a Rigid Body

Conservation of energy in a rotational motion (Section 13.1)

1. A rotating rod

Equations of motion including rotations (Section 13.2)

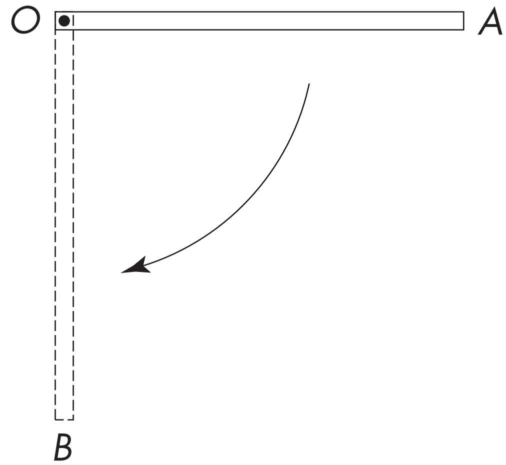
2. A disk and a mass—I
3. A disk and a mass—II
4. Two blocks and a pulley
5. Atwood machine
6. Racing down an incline
7. Equation of motion along an incline

Conservation of angular momentum (Section 13.3)

8. Two objects sliding outward
9. Merry-go-round and a boy

Torque and angular momentum as a vector (Section 13.4)

10. A gyroscope



A uniform rod of length L and mass m is free to rotate about O. The rod is released from state A—that is, from rest in the horizontal position. Define state B to be at the moment when the rod swings past the vertical position. Let the angular velocity of the rod at B be ω. Determine the kinetic energy at state B and the amount of potential energy released from state A to state B.

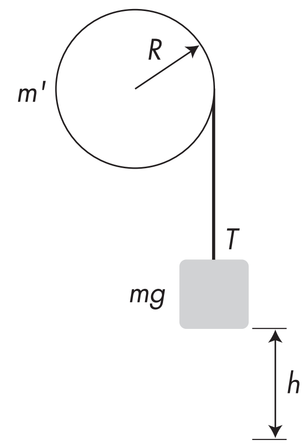
	A	B	C	D
K_B	$mL^2\omega^2/6$	$mL^2\omega^2/24$	$mL^2\omega^2/6$	$mL^2\omega^2/24$
$U_A - U_B$	$mgL/2$	$mgL/2$	mgL	mgL

Hint: Defined with respect to O, $I_{\text{rod}} = mL^2/3$. $U_A - U_B = K_B - K_A$.

Extra: Determine ω at state B.

Explanation: $U_A - U_B = K_B - K_A = K_B$ because $K_A = 0$. The amount of potential energy the rod has released in going from A to B is given by $U_A - U_B = \int gy\,dm = \int gy(dm/dy)dy = (gm/L)\int y\,dy$. Integrating y from 0 to L gives $U_A - U_B = mgL/2$. The kinetic energy of the rod at state B is $K_B = I\omega^2/2 = (mL^2/3)\omega^2/2 = mL^2\omega^2/6$. So Answer = A.

Explanation—extra: The conservation of energy equation implies that $mL^2\omega^2/6 = mgL/2$. Solving for ω gives $\omega = \sqrt{\frac{3g}{L}}$.



A circular disk with mass m' and radius R is mounted at its center, about which it can rotate freely. A light cord wrapped around it supports a block of weight mg. Find the correct equations of motion for the system, where T represents the tension of the cord:

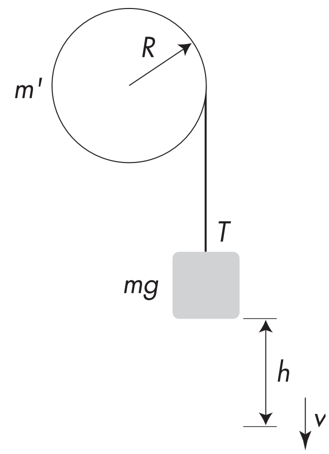
A	$TR = m'R^2\alpha$, and $mg - T = ma$, with $a = R\alpha$
B	$TR = m'R^2\alpha$, and $T - mg = ma$, with $a = R\alpha$
C	$TR = m'R^2\alpha/2$, and $mg - T = ma$, with $a = R\alpha$
D	$TR = m'R^2\alpha/2$, and $T - mg = ma$, with $a = R\alpha$

Extra: Find the block's acceleration a and velocity v after the block, starting from rest, has dropped by a height h. Assume $m' = m$.

Explanation: Remember that for a disk, $I = m'R^2/2$. The equations of motion are $\tau = TR = I\alpha$ and $F = mg - T = ma$. Answer = C.

Explanation—extra: Substituting $I = \dfrac{m'R^2}{2}$ and $m' = m$ into the torque equation gives $T = ma/2$. This leads to $mg - ma/2 = ma$, or $a = 2g/3$. Thus the block is descending with a constant acceleration. It begins from rest. After descending for a distance h, it reaches a speed v. There is the relationship, $v^2 = 2(2g/3)h$, or $v = \sqrt{\dfrac{4gh}{3}}$.

2. A Disk and a Mass—I



Consider the same setup as in the previous question. A circular disk with mass m' and radius R is mounted at its center, about which it can rotate freely. A light cord wrapped around it supports a block of weight mg. Find the total kinetic energy of the system at the moment when the block is at speed v:

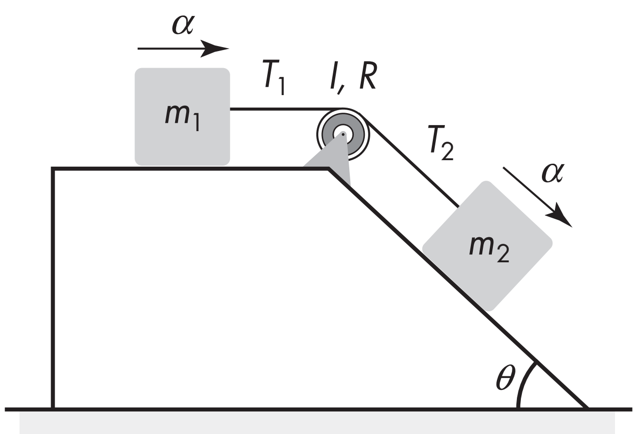
A	B	C	D
$mv^2/2$	$3mv^2/4$	mv^2	$5mv^2/4$

Extra: Based on conservation of energy, express v in terms of the distance fallen, h.

Explanation: Using $v = m\omega$,

$K = K_{trans} + K_{rot} = mv^2/2 + mR^2\omega^2/4 = 3mv^2/4$. Answer = B.

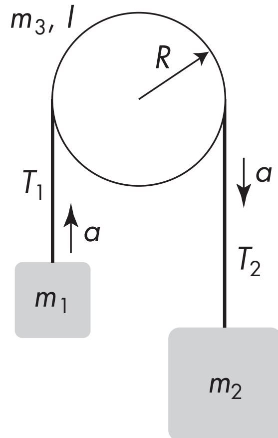
Explanation—extra: Denote the initial state by A, where there is no kinetic energy (that is, $K_A = 0$), and the state after falling through a height h by B. Conservation of energy implies that $K_B - K_A = K_B = U_A - U_B = mgh = 3mv^2/4$. Or $v = \sqrt{(4gh/3)}$.



Blocks with masses m_1 and m_2 are connected by a string, which passes over a pulley. The pulley has a radius R and moment of inertia I. The acceleration of the two masses is a, and the pulley is constrained to rotate with an angular acceleration $\alpha = a/R$. Denote the tension in the rope at block m_1 to be T_1 and at block m_2 to be T_2. The rotational equation of motion of the pulley is given by which of the following?

A	$(T_1 - T_2)R = Ia/R$
B	$(T_2 - T_1)R = Ia/R$
C	$(T_1 - T_2)R = Ia$
D	$(T_2 - T_1)R = Ia$

Explanation: The rotational equation of motion is given by $\tau = I\alpha = Ia/R$. Because block m_2 is descending and the pulley is rotating clockwise, T_2 is greater than T_1. Answer = B.

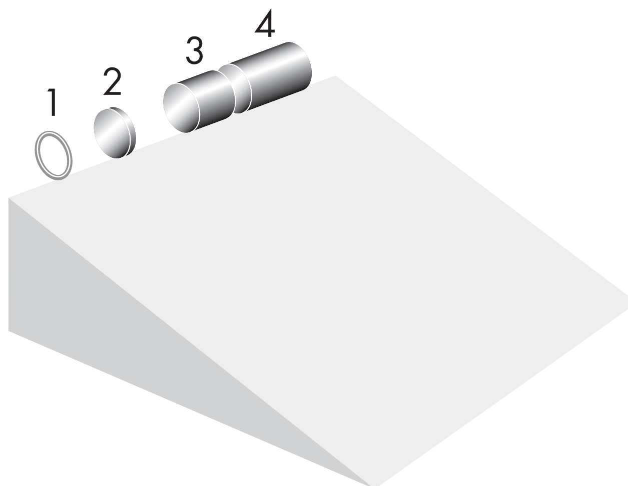


Consider the Atwood machine shown in the sketch. The pulley has a radius R, the moment of inertia of the pulley is I, and $m_2 > m_1$, so that the block of mass m_2 is moving *down* with acceleration a. Because there are three moving objects, there should be three equations of motion. Two of them are $T_1 - m_1 g = m_1 a$ and $m_2 g - T_2 = m_2 a$. What is the third equation?

A	$(T_1 - T_2)R = Ia/R$
B	$(T_2 - T_1)R = Ia/R$
C	$(T_1 - T_2)R = Ia$
D	$(T_2 - T_1)R = Ia$

Explanation: The equation of motion for the pulley is $\tau = I\alpha$, where $\alpha = a/R$. Notice that because m_2 is descending, T_2 should be greater than T_1. Answer = B.

Consider a race in which the four objects with properties listed here roll down an inclined plane:



		Mass	Radius
#1	Ring	m	R
#2	Disk	m	R
#3	Solid cylinder 1	$m/2$	$R/4$
#4	Solid cylinder 2	$m/4$	$R/4$

Choose the correct set:

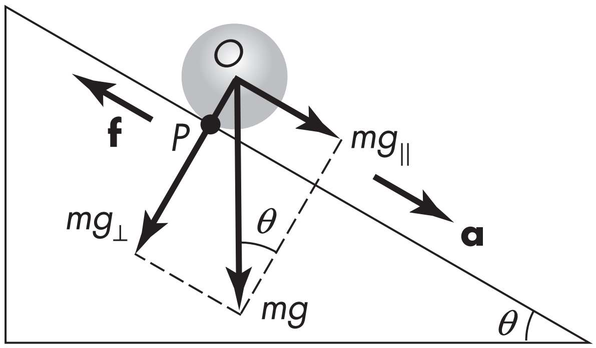
	A	B	C	D
Fastest	#2 only	#2 only	#2 only	#2, 3, 4
Slowest	#1	#4	#3, 4	#1

Hint: $I_{ring} = mR^2$, $I_{disk} = I_{solid\text{-}cyl} = mR^2/2$. Denote $k = I/mR^2$. $mgh = mv_i^2/2 + I_i\omega_i^2/2 = (mv_i^2/2)[1 + k_i]$, with $k_i = [I/(mR^2)]_i$. Here $i = 1, 2, 3, 4$.

Extra: Find v in terms of h for #2.

Explanation: From the hint, $k_1 = 1$ and $k_2 = k_3 = k_4 = 0.5$. Because $mgh = \left(\frac{mv_i^2}{2}\right)[1 + k_i]$, then the smaller k is, the greater is the speed. So #2, #3, and #4 are equally the fastest, and #1 is the slowest. Answer = D.

Explanation—extra: $k_2 = 0.5$. So $mgh = (mv^2/2)[1 + 0.5] = (3/4)mv^2$, or $v = \sqrt{\frac{4gh}{3}}$.



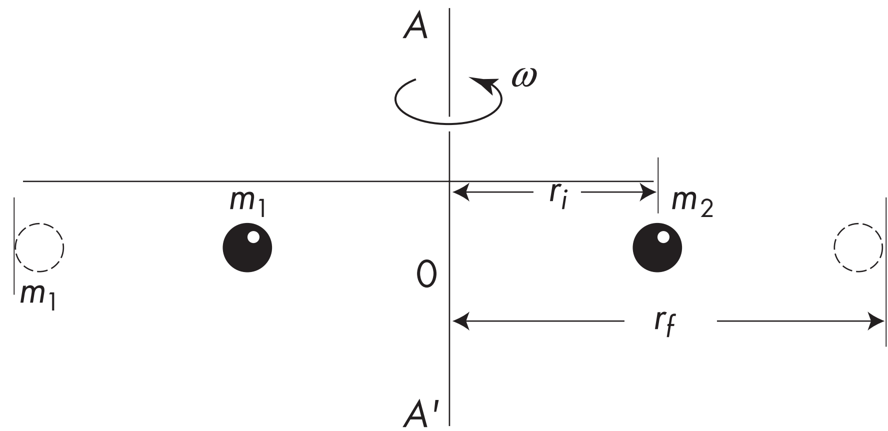
A	$mg_{\|}R = kmR^2\ \alpha$
B	$mg_{\|}R = (1 + k)mR^2\ \alpha$
C	$mgR = kmR^2\ \alpha$
D	$mgR = (1 + k)mR^2\ \alpha$

Consider a ball of mass m with a moment of inertia about its center O of $I_o = kmR^2$, rolling down an incline. The angle between the incline and the horizontal direction is θ. The equation of motion along the incline, $\tau = I\alpha$, is given by which of the following?
Hint: Evaluate the torque about P, the contact point.

Extra: Show that $a = \dfrac{g \sin \theta}{1 + k}$. From $mg_{\|} - f = ma$, show that for $\theta > \theta_c$, the ball will slip, where **tan** $\theta_c = \dfrac{\mu(1 + k)}{k}$ and μ is the coefficient of static friction between the ball and the incline.

Explanation: By inspection, the torque about P is $\tau = mg_{\|}R$. Using the parallel axis theorem, the moment of inertia of the ball about P is $I_P = (1 + k)mR^2$. So the equation of motion now reads $mg_{\|}R = I_P\alpha = (1 + k)mR^2\alpha$ (1). Answer = B.

Explanation—extra: Using $\alpha = \dfrac{a}{R}$ and $mg_{\|}R = mgR \sin \theta$ in Eq. (1) gives $mgR \sin \theta = (1 + k)mRa$. Solving for the acceleration given $a = \dfrac{g \sin \theta}{1 + k}$. The critical point occurs when the friction force is the maximum possible between the ball and the incline: $f = f^s{}_{max} = \mu mg_{\perp}$. Substituting f and a into $mg_{\|} - f = ma$ gives $mg \sin \theta -$ cmg $\cos \theta = mg \sin \theta$ liter. Solving for tan θ gives the required result.



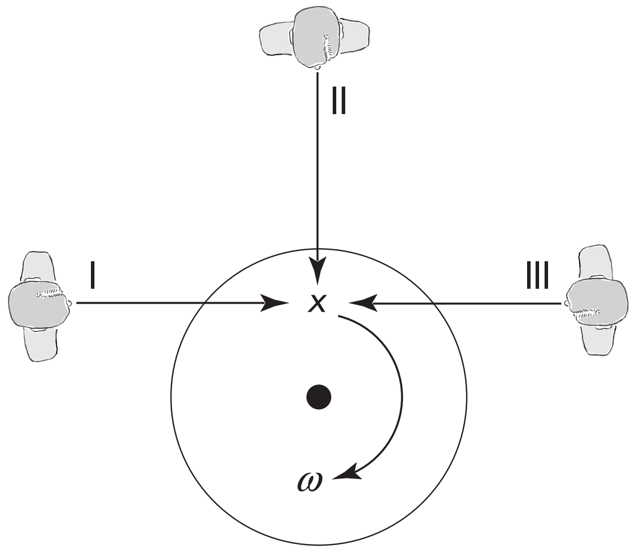
Two small objects of masses m_1 and m_2 are initially fixed at $r_i = R/2$ on either side of axis AA'. They are rotating about the axis AA' with an angular velocity ω_i. Then they are released from $r = r_i$ and end up at $r_f = R$. Assume the process of releasing the objects does not change the angular momentum. Determine their new angular velocity ω_f:

	A	B	C	D
ω_f	$4\omega_i$	$2\omega_i$	$\omega_i/2$	$\omega_i/4$

Hint: Conservation of angular momentum here implies that $I_i\omega_i = I_f\omega_f$.

Explanation: $I_i\omega_i = I_f\omega_f$ implies that $\omega_f = (I_i/I_f)\omega_i$. But $I = \Sigma mr^2$. When r is doubled, I is increased by a factor of 4. This leads to $\omega_f = \omega_i/4$.
Answer = D.

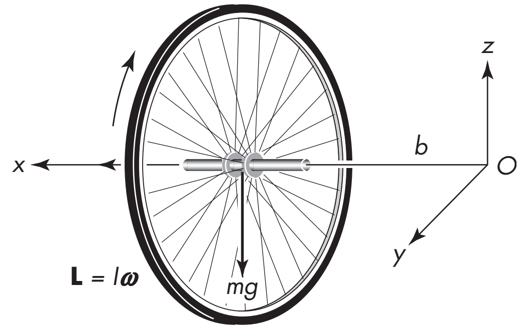
The sketch shows the top view of a merry-go-round. It is rotating clockwise. A boy is jumping onto the merry-go-round in three different ways: (I) from the left, (II) from the top, and (III) from the right. For all three cases, the boy lands on the same spot. Compare final angular moments and momenta of inertia of the final system for these cases:



	L_{final}	I_{final}
A	$L_I > L_{II} > L_{III}$	$I_I = I_{II} = I_{III}$
B	$L_I > L_{II} > L_{III}$	$I_I > I_{II} > I_{III}$
C	$L_I = L_{II} = L_{III}$	$I_I = I_{II} = I_{III}$
D	$L_I = L_{II} = L_{III}$	$I_I > I_{II} > I_{III}$

Hint: Use vector addition of angular momentum vectors.

Explanation: With respect to the center of the merry-go-round, the angular momentum of the boy is (I) clockwise, (II) zero, and (III) counterclockwise. The angular momentum of the merry-go-round is clockwise. The addition of the angular momentum of the boy in each case to that of the merry-go-round leads to $L_I > L_{II} > L_{III}$. Notice that for all three cases the boy lands at the same distance from the center. This implies that the momenta of inertia of the boy plus the merry-go-round are the same for all three cases—that is, $I_I = I_{II} = I_{III}$. Answer = A.



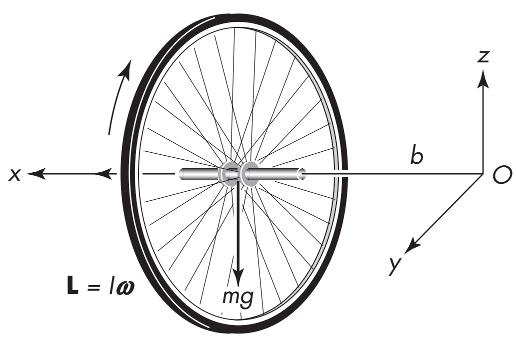
In this setup a bicycle wheel is rotating about a horizontal axis. At the moment shown, the angular momentum is pointing along the *x* axis. Determine the direction of precession about *O* as viewed from the top:

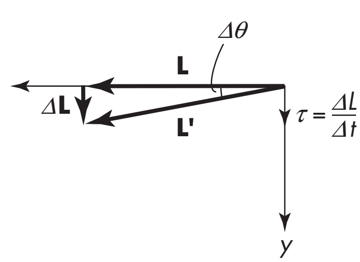
A	Clockwise
B	Counterclockwise

Hint: The torque about *O* due to the weight of the wheel is along the positive *y* direction. This implies that the change of **L**, $\Delta \mathbf{L} = \boldsymbol{\tau} \Delta t$, is along the positive *y* direction.

Extra: What is the angular velocity of precession about *O* if the distance from *O* to the wheel's center is *b*?

Explanation: As viewed from the top, the increment $\Delta \mathbf{L}$ is in the positive *y* direction, which leads to a counterclockwise motion. Answer = B.





Explanation—extra: For the precession's angular velocity, $\omega_{\text{precession}} = \dfrac{\Delta\theta}{\Delta t} \approx \dfrac{\Delta L}{L\Delta t} = \dfrac{\tau}{I\omega} = \dfrac{mgb}{I\omega}$.