SHERWOOD THEORY MEETING 1988

The Magnetic Field Lines on a Perfect Conductor or Circle Maps Generated by the Neumann Problem in a Torus

P.J. Morrison and J.D. Meiss
Institute for Fusion Studies
The University of Texas at Austin
Austin, Texas 78712

The generic behavior that occurs in maps of the circle to itself serves as a unifying model for nonlinear physical systems with two competing periods. It has successfully modelled quasiperiodic, mode-locking, and chaotic phenoma in a variety of physical situations ranging from Josephson junctions to geophysics. A circle map is naturally generated by solving Laplaces equation in a topologically toroidal region with Neumann boundary conditions. The Neumann problem yields the magnetic field in a toroidal current free region, from which the field lines at the boundary are determined. These field lines define a map of the closed curve, defined by a poloidal cut of the boundary, to itself. For systems with symmetry separable solutions are easily obtained; these solutions are not particularly interesting. In the case without symmetry de Rham's theorem guarantees solutions with arbitrary periods; i.e. net toroidal and poloidal currents. We will discuss the kinds of solutions that can occur and their physical relevance.

poster

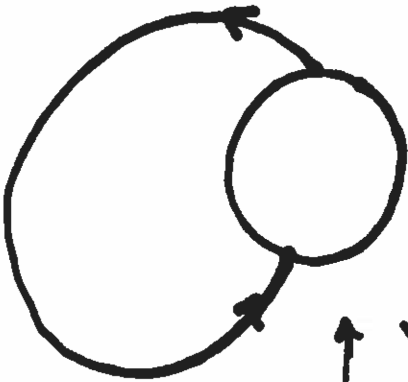
Gatlinburg

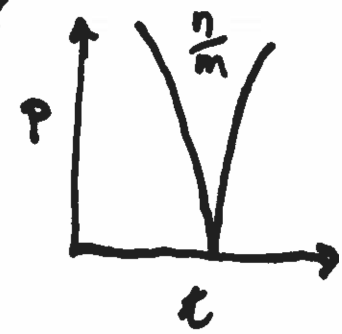
Field Lines on a Conducting Surface

Circle Maps

&

Mode Locking

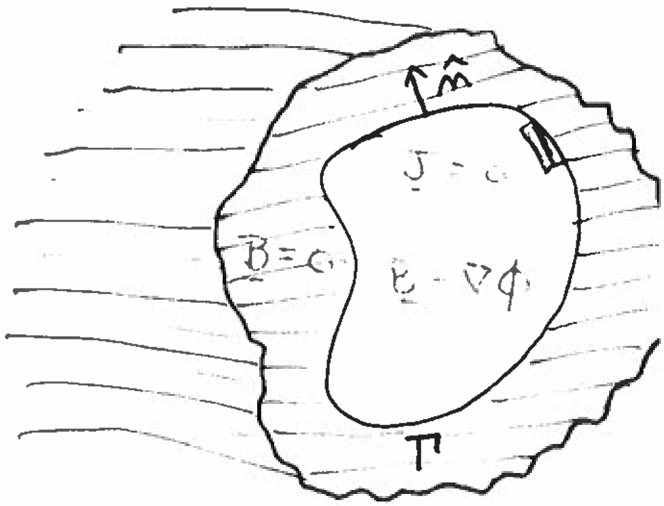




J.D. Meiss
P.J Morrison

Inst. Fusion Studies
Univ. of Texas

What are the field lines like on the surface of a toroidal perfect conductor?



Neumann Problem

$$\underline{J} = 0 \quad\quad \Rightarrow \quad \nabla^2 \phi = 0$$
$$\underline{B} = \nabla \phi$$

$$\left.\frac{\partial \phi}{\partial n}\right|_S = 0 \qquad S = \text{inner surface}$$

$\Rightarrow$

$$\oint_\Gamma \underline{B} \cdot d\underline{\ell} = I_T$$

$$\oint_\gamma \underline{B} \cdot d\underline{\ell} = I_P$$

The Neumann problem defines a circle map.

$$\Gamma \longrightarrow \Gamma$$

Circle Maps

A model for complex systems of nonlinearly coupled oscillators

E.G. Damped driven pendulum

Semiconductor physics

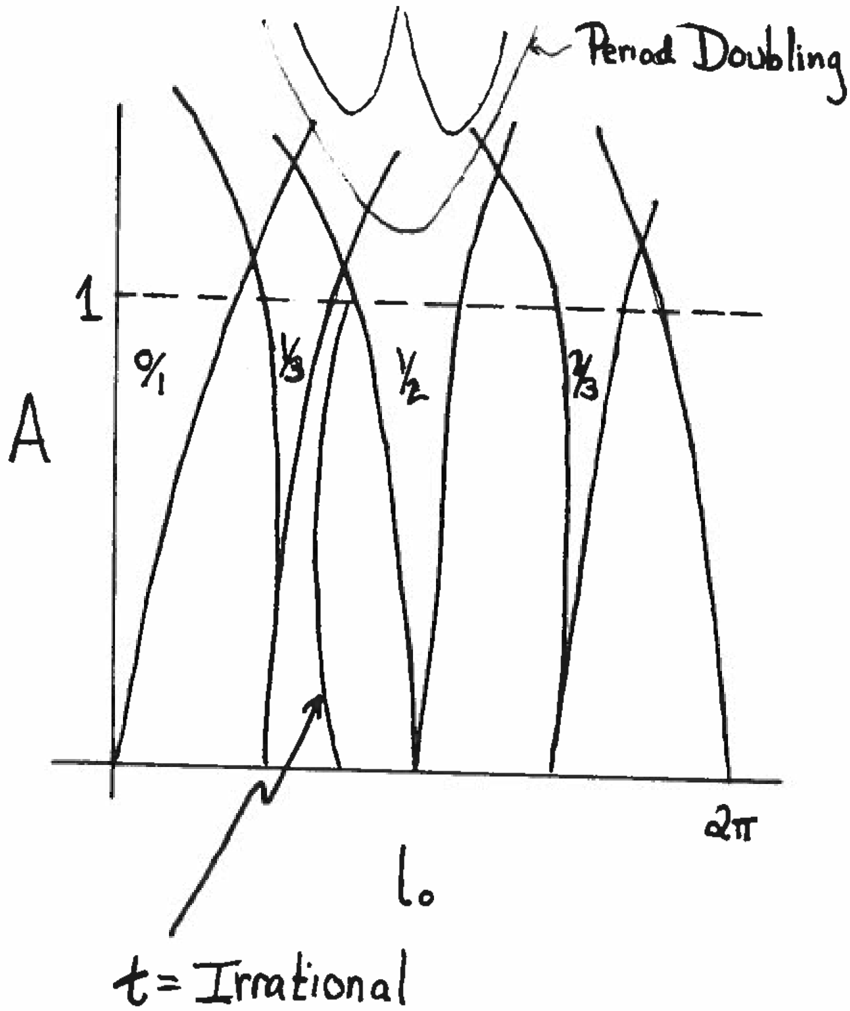
Josephson junctions

Rayleigh-Benard convection

Describes mode locking and the transition to chaos

Sine Circle map

$$\theta_{n+1} = \theta_n + 2\pi t_0 + A \sin \theta_n$$


Period Doubling
1
0/1
1/3
1/2
2/3
A
2π
l₀
t = Irrational

Circle Maps : Generic Behavior of systems with two frequencies

1) Trivial Case $(A=0)$ $\quad t = t_0$

uniform rotation with rational or irrational rotational transform

2) Monotone $(0<A<1)$ $\quad t = \frac{1}{2\pi n}\,\theta(n)$ as $n \to \infty$

All orbits have definite rotational transform

$\Rightarrow$ rate is not uniform

Two cases

i) Irrational - occurs for single curve in parameter space

ii) Rational - occurs for intervals

Mode Locking $\quad t = n/m$

3) Non-Monotone $(A>1)$

Chaos, period doubling, etc.

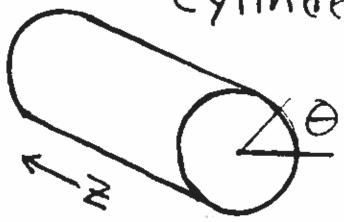
Possibilities

$\iota \equiv$ rotational transform

I. $\iota = 0 \Leftrightarrow$ identity map

Ideal torus or right circular periodic cylinder.

$$\Phi(r,\theta,z) = J_z z + J_\theta \theta$$

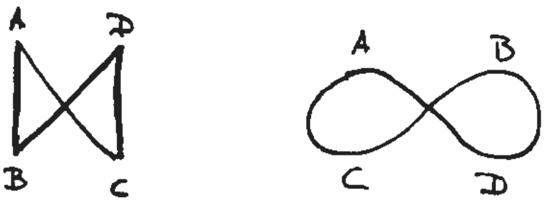


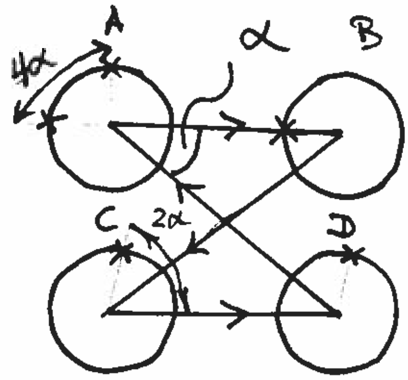
$$\underline{B} = J_z \hat{z} + \frac{\hat{\theta}}{r} J_\theta$$

regularity at $r=0 \Rightarrow J_\theta = 0 \Leftrightarrow$ stellarator

II. $\iota =$ rational or irrational $\Leftrightarrow$ $\theta(n)$ fixed point map or ergodic map

Figure-eight stellarator



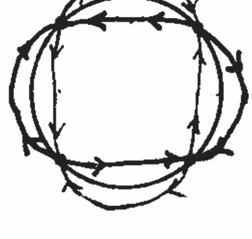


$\underline{L = 4\alpha}$

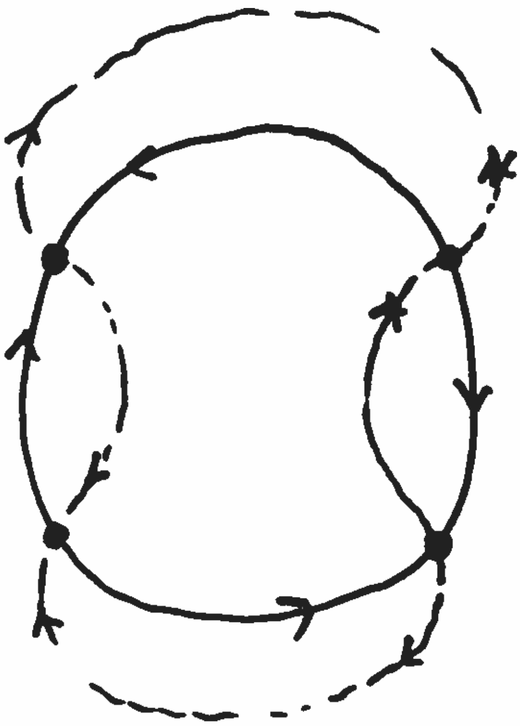
III. $\iota =$ rational with <u>attracting</u> fixed points

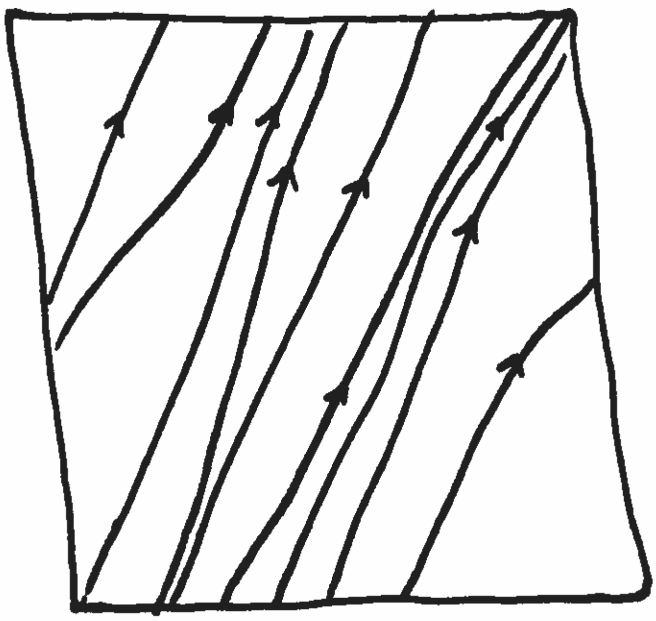
Separatrix surface

$$r = a\left(1 + \varepsilon \cos(2\theta - nz)\right)$$



$\ell = 2$





Mode Locking Surfaces have 2 closed lines

Wavy Periodic Cylinder

Separation solution: $\phi(r,\theta,z) = \alpha z + \sum_{l,n} I_l(hr) \begin{Bmatrix} \sinh hz \\ \cosh hz \end{Bmatrix} \begin{Bmatrix} \sin l\theta \\ \cos l\theta \end{Bmatrix}$

Boundary w/ perturbation: $r(\theta,z) = 1 + \varepsilon f(\theta,z)$ $\varepsilon <<$

$\Rightarrow$ arbitrary

$$\hat{n} = c\left(\hat{r} - \varepsilon \hat{z}\frac{\partial f}{\partial z} - \frac{\varepsilon\hat{\theta}}{r}\frac{\partial f}{\partial \theta}\right)$$

Boundary condition: $\hat{n}\cdot\nabla\phi = \sum_{\alpha=0} \varepsilon^{\alpha}\Big[\sum_{l,n=0} h I_l'(hr(\theta,z))\, R^{(\alpha)}(hz,l$

$$- \varepsilon r(\theta,z)\frac{\partial f}{\partial z}\left\{\alpha + \sum_{l,n=0} h I_l(hr)\, R_1^{(\alpha)}(hz,l\theta)\right\}$$

$$- \varepsilon \frac{\partial f}{\partial \theta}\sum_{l,n=0} l\, I_l(h,z)\, R_2^{(\alpha)}(hz,l\theta)\Big] =$$

where $R(hz,l\theta) = a_{n,l}\cosh hz \cos l\theta + b_{n,l}\cosh hz \sin l\theta$
$+ c_{n,l}\cos l\theta \sinh hz + d_{n,l}\sinh hz \sin$

Boundary coefficients: $f(\theta,z) = \sum_{n,l=0}\big[A_{n,l}\cosh hz\cos l\theta + B_{n,l}\cosh hz \sin l$

$+ C_{n,l}\cos l\theta\sinh hz + D_{n,l}\sinh hz\sin l$

Order ε

matching to zero : $\sum_{h,\ell} h\, I_\ell'(h)\, \mathcal{A}^{(1)}(hz, \ell\theta) = \alpha \frac{\partial f}{\partial z}$

$\Longrightarrow$

$$\phi(r,\theta,z) = \alpha z + \varepsilon\alpha \sum_{h,\ell=0} \frac{I_\ell(hr)}{I_\ell'(h)} \Big\{ -A_{h,\ell} \cos\ell\theta \sin\ldots$$

$$-B_{h,\ell} \sinh hz \sin\ell\theta + D_{h,\ell} \cosh hz \cos\ell\theta + C_{h,\ell} \cosh hz \sin\ldots$$

field:

$$\vec{B} = \varepsilon\alpha \frac{\partial f}{\partial z}\hat{r} + \varepsilon\alpha\hat{\theta} \sum_{\ell,h=0} \ell \frac{I_\ell(h)}{I_\ell'(h)} \Big\{ -D_{\ell h} \cos\ldots$$

$$+ \hat{z} \Big[\alpha - \varepsilon\alpha \sum_{h,\ell} h \frac{I_\ell(h)}{I_\ell'(h)} \{ \cdots \} \Big]$$

field lines: $\frac{B_z}{dz} = \frac{B_\theta}{d\theta} \Longrightarrow$

$$\frac{d\theta}{dz} = \varepsilon \sum_{h,\ell} \ell \frac{I_\ell(h)}{I_\ell'(h)} \Big\{ A_{h,\ell} \sin\ell\theta \sinh hz + \cdots \Big\}$$

Single Mode:

$$\frac{d\theta}{dz} = a \cos(\ell\theta - hz)$$

Single Mode Analysis

$$\frac{d\theta}{dz} = \omega + a\cos(m\theta - nz)$$

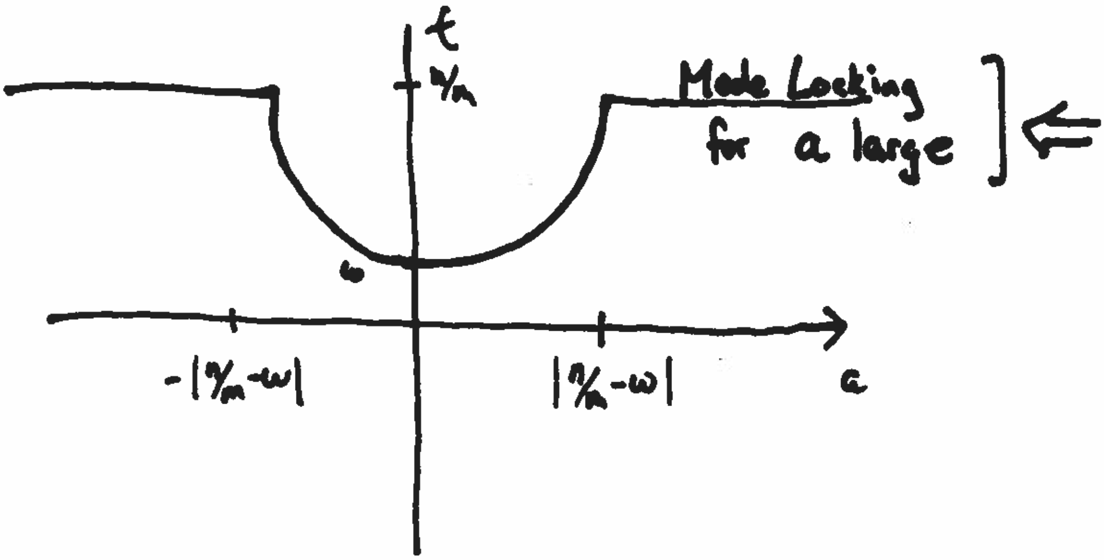
Let $\bar{\theta} = \theta - {}^{n}\!/_{m}\, z$ (Helical Angle)

$$\frac{d\bar{\theta}}{dz} = (\omega - {}^{n}\!/_{m}) + a\cos m\bar{\theta}$$

Circle Map obtained by $\theta(z=1) = \int_0^1 \frac{d\theta}{dz}(\theta, z)$

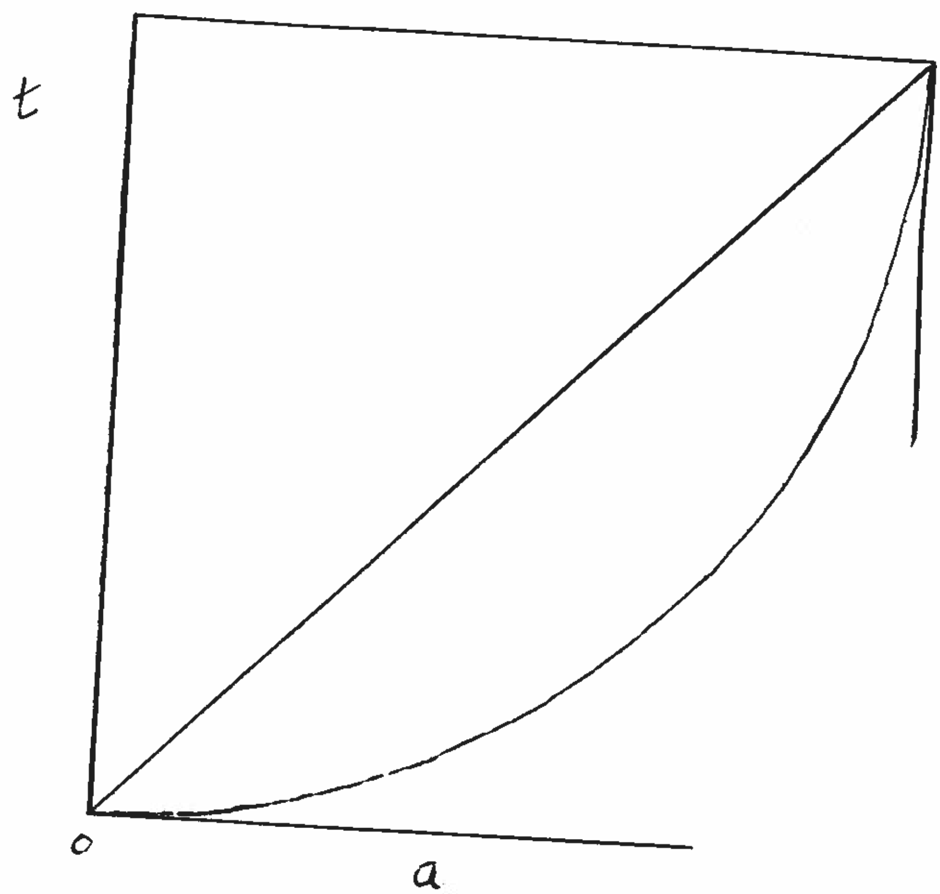
$\Longrightarrow$ rotational transform

$$t = \omega + \left(\frac{n}{m} - \omega\right)\left[1 + \sqrt{1 - \left(\frac{ma}{n - m\omega}\right)^2}\,\right]$$



Rotational transform from orbit Integrat

$\omega = 0$



$$ι = \left\langle \frac{\Delta\theta}{\Delta z} \right\rangle$$

Several Mode Case

Use Averaging Perturbation Theory

$$\dot{\theta} = \omega + \varepsilon f(\theta, t)$$

Let $\alpha = \theta - \omega t$

$$\dot{\alpha} = \varepsilon f(\alpha - \omega t, t) = g(\alpha, t)$$

Assume

Fluctuating part

$$\alpha = C(t) + \sum_{j=1}^{\infty} \varepsilon^j A^j(c,t)$$

Slowly Varying Part

$$\dot{c} = \sum_{j=1}^{\infty} \varepsilon^j C^j(c,t)$$

Each order in ε get

$$C^j + \frac{\partial A^j}{\partial t} = \Pi(\text{Lower order terms})$$

choose $C^j = \frac{1}{T}\int_0^T dt\,(\downarrow)$ for fixed c

resonant denominator

Single Mode

$$t = \dot{c} = \underbrace{-\frac{1}{2}\frac{ma^2}{(m\omega-n)} - \frac{1}{8}\frac{m^3a^4}{(m\omega-n)^3}}_{\text{expansion of exact result}} + \mathcal{O}(a^6)$$

Two Mode

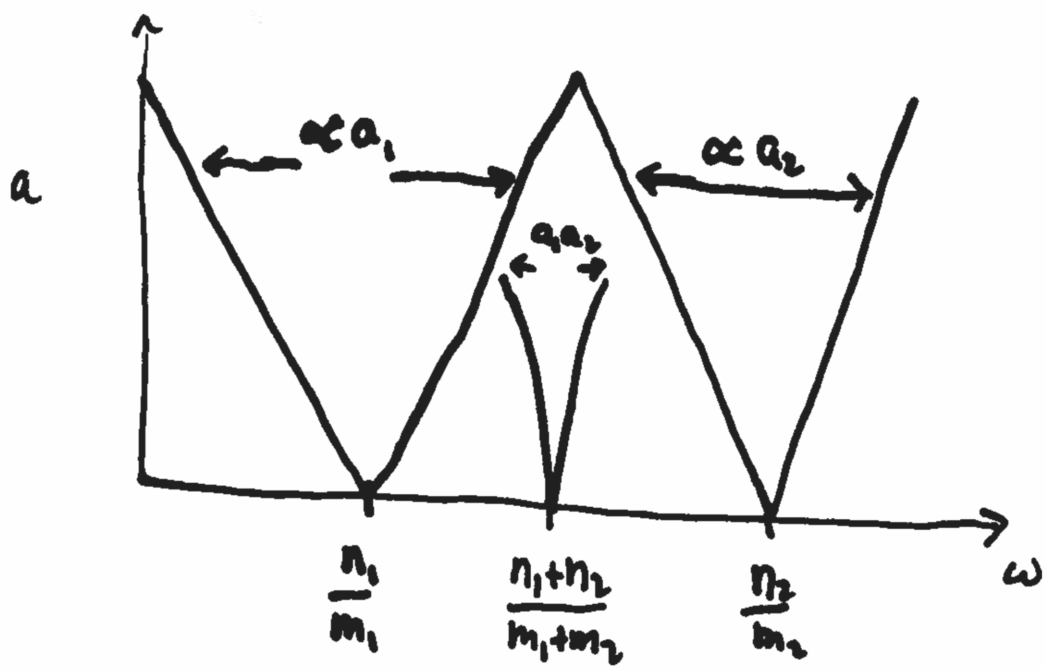
$$t = \dot{c} = -\frac{1}{2}\frac{m_1 a_1^2}{(m_1\omega_1 - n_1)} - \frac{1}{2}\frac{m_2 a_2^2}{m_2\omega_2 - n_2} + \text{Interactio}$$

Arises from $\frac{1}{T}\int_0^T \cos(m_1\theta_1 - n_1 z)\cos(m_2\theta_2 - n_1$

Non-zero only for $\omega = \frac{n_1 \pm n_2}{m_1 \pm m_2}$

Farey Resonance

Arnold Tongues to $2^{\underline{nd}}$ Order

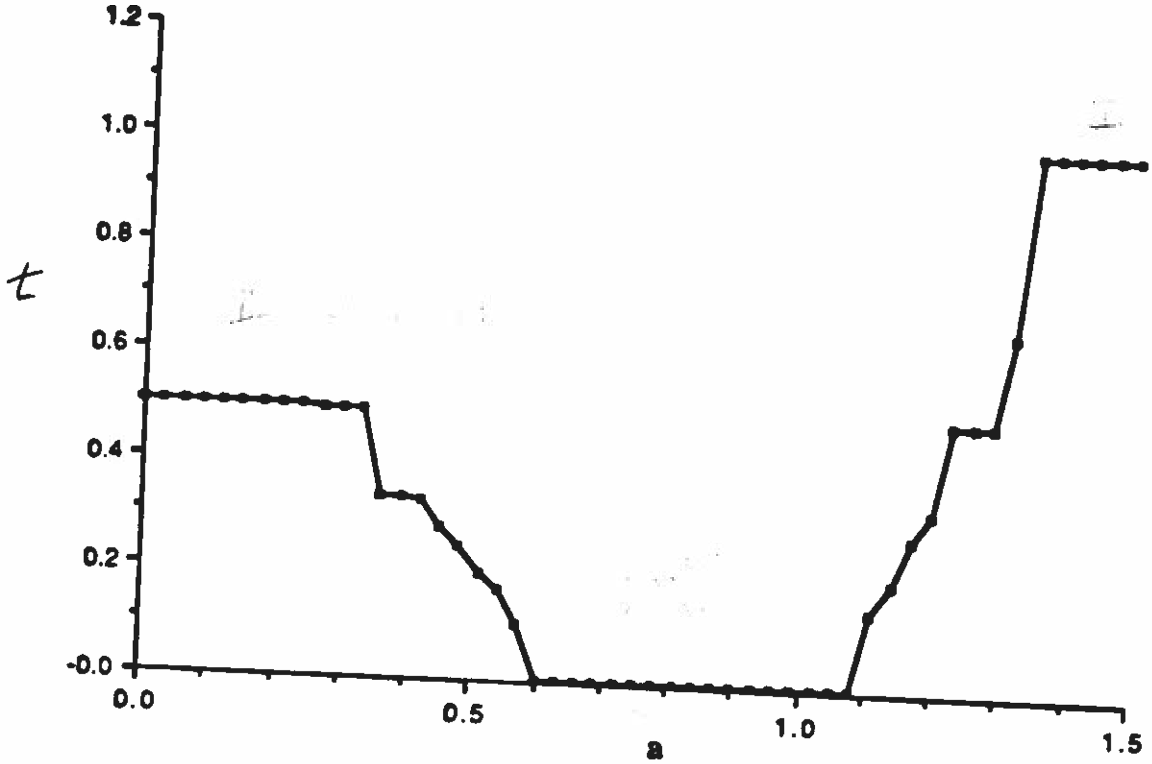


$2^{\underline{nd}}$ Order Tongue Determined by

$$\frac{1}{2} a_1 a_2 \left[\frac{m_1}{m_2\omega - n_2} + \frac{m_2}{m_1\omega - n_1}\right] > \Bigg| \delta\omega - \frac{1}{2}\left[\frac{m_1 a_1^2}{m_1\omega - n_1} + \frac{m_2}{m_2\omega}\right.$$

gives quadratic tongue

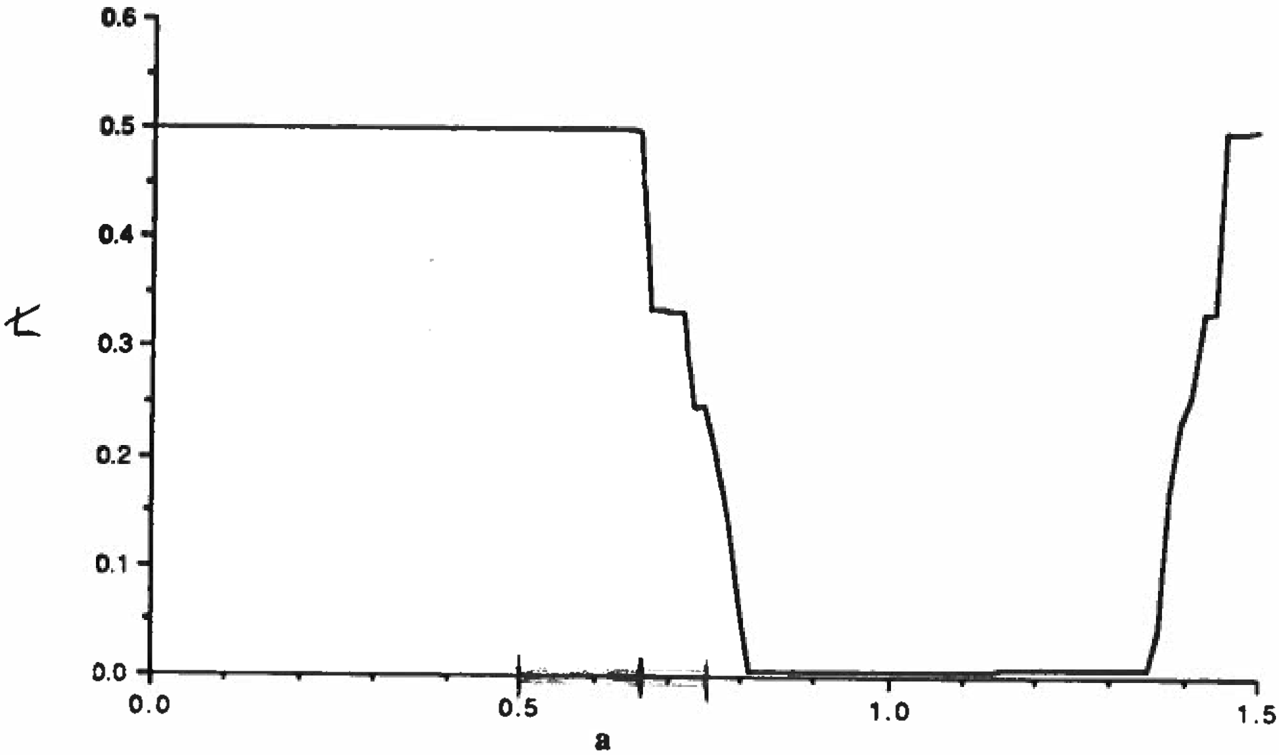
$$\frac{d\theta}{dz} = a\cos(\theta - z) + (.6)\cos(2\theta - z)$$



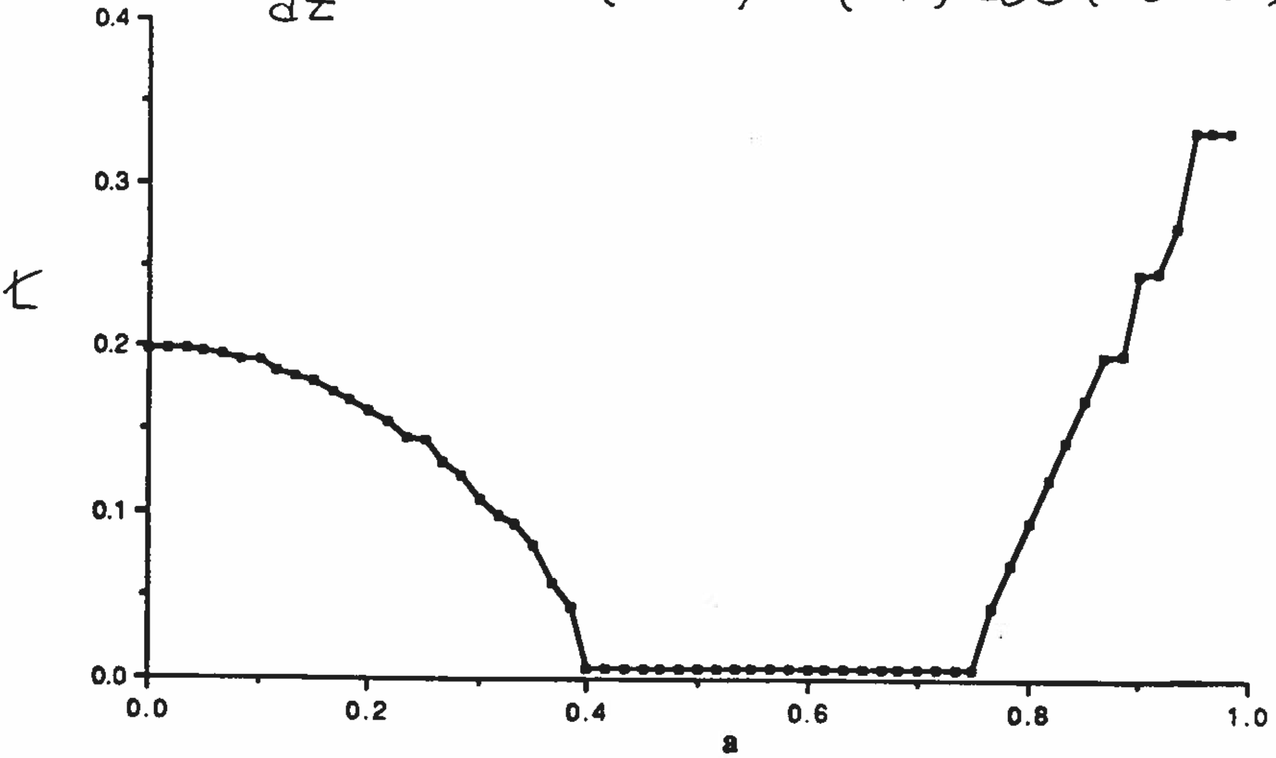
ι = Rotational Transform

$\bar{\iota} = \iota / 2\pi$

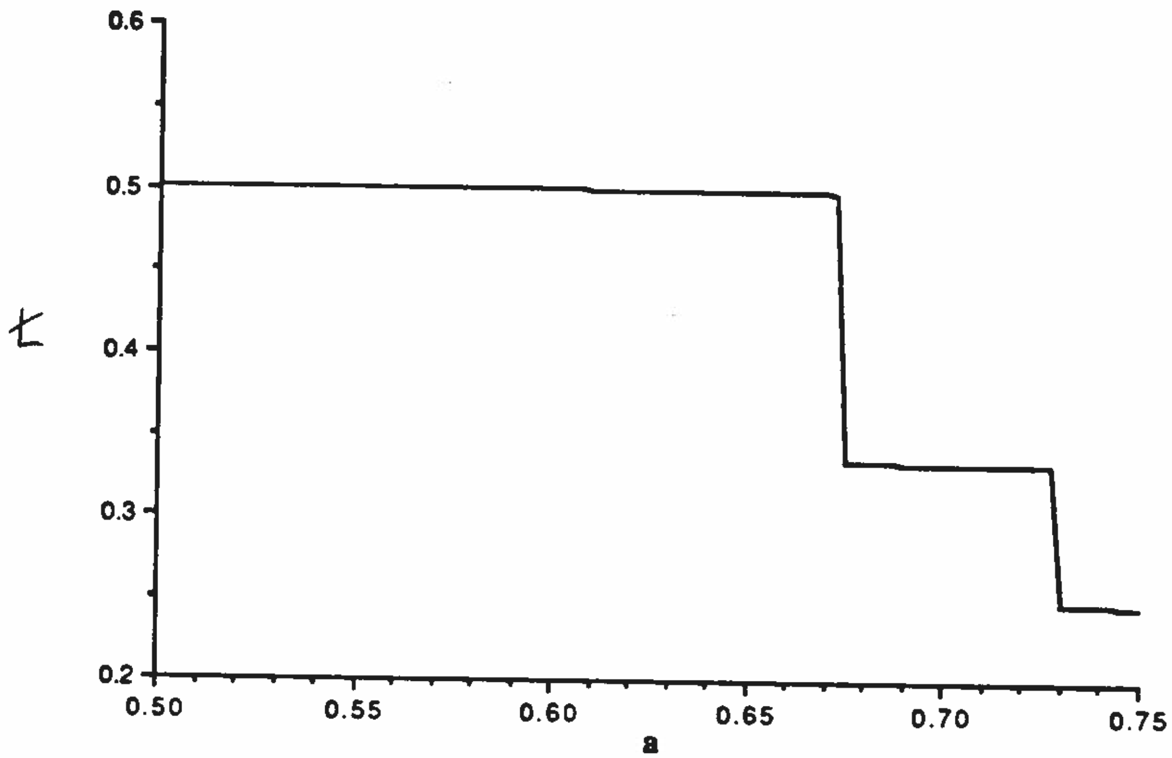
$$\frac{d\theta}{dz} = a\cos(\theta - z) + (.8)\cos(2\theta - z)$$



$$\frac{d\theta}{dz} = a\cos(\theta - z) + (.4)\cos(2\theta - z)$$

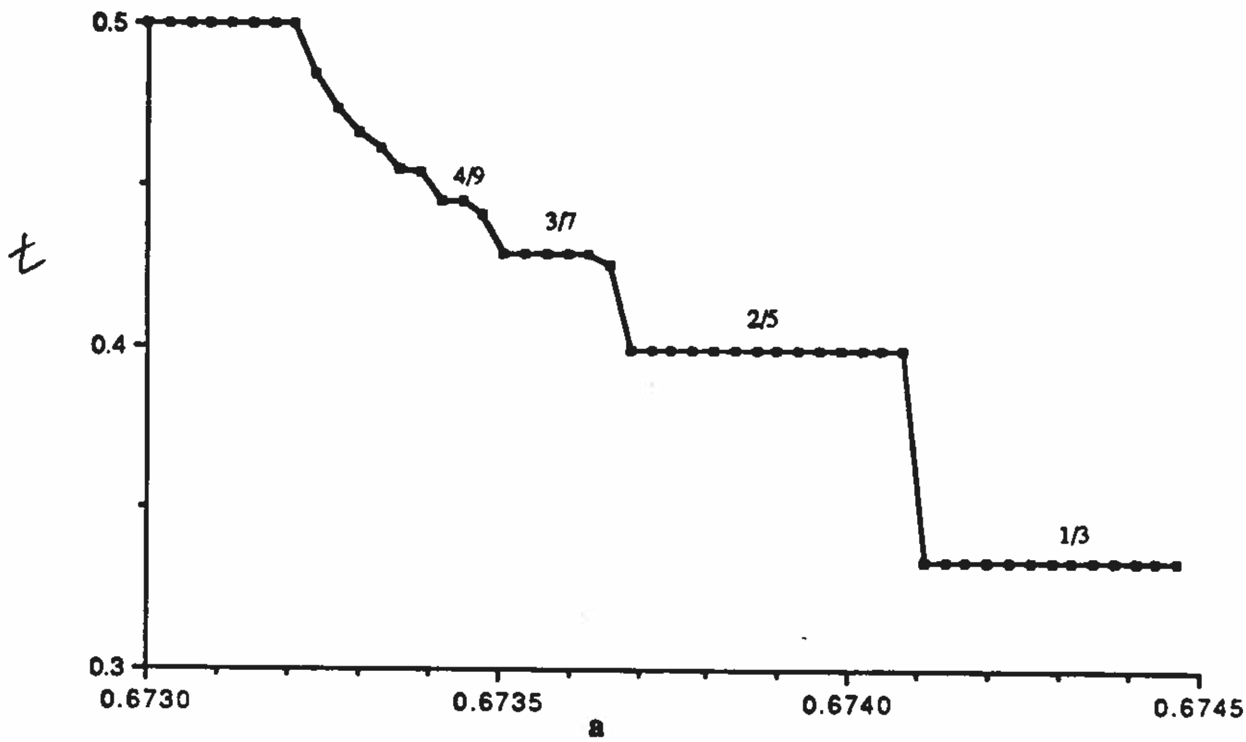


$$\frac{d\theta}{dz} = a\cos(\theta - z) + (.8)\cos(2\theta - z)$$



← BLOW UP →

$$\frac{d\theta}{dz} = a\cos(\theta - z) + (.8)\cos(2\theta - z)$$



← BLOW UP AGAIN →

Two Mode Staircases

Sine Circle Ode

$$\frac{d\theta}{dz} = \frac{B_\theta}{B_z}$$

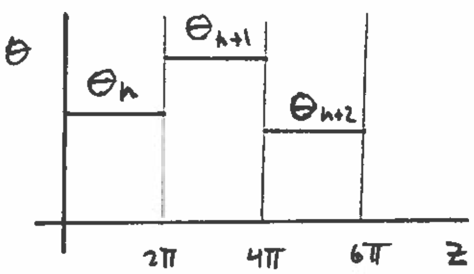
$$= a \sum_{n=0}^{\infty} \cos(nz) + b \sum_{n=0}^{\infty} \left[\cos(\theta - nz) + \cos(\theta + nz) \right)$$

kick at $z \equiv 0$

Standing wave superposition

$$\sum_{n=-\infty}^{\infty} e^{int} = 2\pi \sum_{n=-\infty}^{\infty} \delta(t - 2\pi n) \implies$$

$$\frac{d\theta}{dz} = 2\pi (a + b \sin\theta) \sum_{n=-\infty}^{\infty} \delta(z - 2\pi n)$$



$$\int_{0^+}^{2\pi^+} \frac{d\theta}{dz}\, dz \implies$$

$L = 2\pi a$

$A = 2\pi b$

$$\theta_{n+1} = \theta_n + i_o + A \sin\theta_n$$

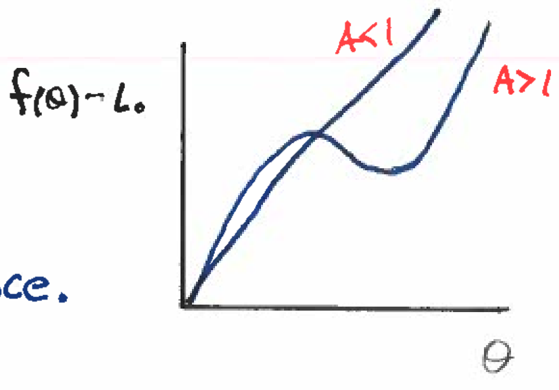
Behavior of Sine Circle Map

$A = 0$: uniform rotation — "tokamak-like" rational or irrational surfaces ($\iota_0 \in \emptyset$?)

$0 < A < 1$: mode locking — orbit asymptotically attracts to asym periodic fixed point of rotational transform, ι.

$A = 1$: Critical value for start of transition to chaos. Start of logistic map behavior.

$A > 1$: Beginning of period doubling bifurcation sequence.



Single Mode Analysis

field line equation: $$\frac{d\theta}{dz} = a\cos(\ell\theta - hz)$$

integration scaling: $$t_0 \equiv h/\ell \qquad \bar{\theta} = \theta - t_0 z$$

$$\frac{d\bar{\theta}}{dz} = -t_0 + a\cos \ell\bar{\theta}$$

Circle map:

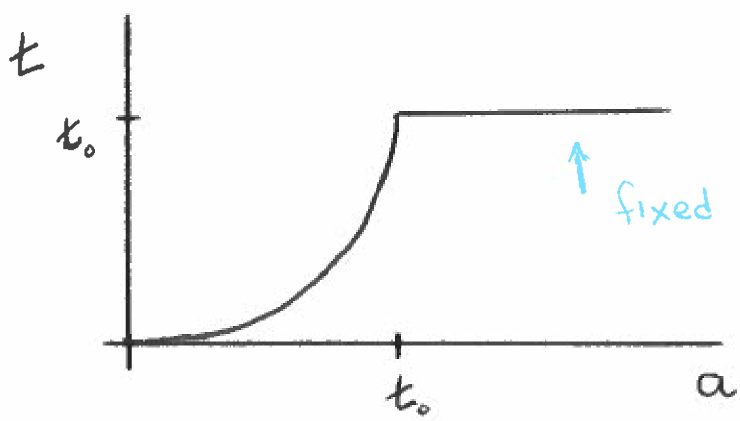
$$\frac{h}{2}(z - z_0) = \frac{t_0}{a + t_0}\int_{\tan\left(\frac{\ell\theta_0 - hz_0}{2}\right)}^{\tan\left(\frac{\ell\theta - hz}{2}\right)} \frac{dx}{\left(\frac{a - t_0}{a + t_0}\right) - x^2}$$

two branches:

$a < t_0$

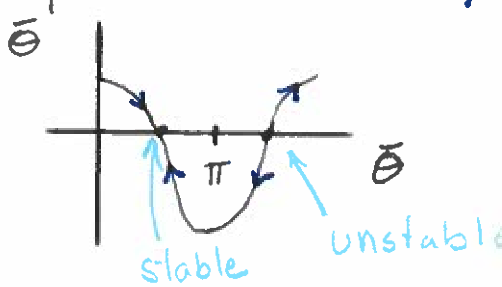
$$t = \frac{\Delta\theta}{\Delta z} \Rightarrow$$

$$t = t_0 - \sqrt{t_0^2 - a^2}$$



$a > t_0$

$$\frac{d\bar{\theta}}{dz} = 0$$ Asymptotica[lly]
Stable equ[ilibrium]



$$\Rightarrow t = t_0$$